MATH 13 **EXAM #1** **NAME:** Key

Use both sides of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your logic and ultimately your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline and mathematical rigor that was illustrated in the lecture for the proofs.

1. If V is a vector space, prove that $0 \odot \hat{v} = \hat{0}$. (*Justify all steps as illustrated in the lecture*)

2. Given $S = \{\hat{v}_1, \ldots, \hat{v}_k\} \subseteq V$ a vector space. If at least one vector in S can be written as a linear combination of the rest of the vectors in S, then prove that S is a linearly dependent set.

3. Given V is a vector space, $B = \{\hat{v}_1, \ldots, \hat{v}_n\}$ is a basis of V and $S = \{\hat{u}_1, \ldots, \hat{u}_m\} \subseteq V$. Prove that if $m > n$, then S is a linearly dependent set.

4. Given $A_{m\times n}$. Prove that $N(A)$ is a subspace of $\mathbb{R}^{\square}$ (fill-in the blank and then prove).

5. Given $\dim(V) = n$ (V is a vector space). If $B = \{\hat{v}_1, \ldots, \hat{v}_n\}$ is linearly independent, then prove that B is a basis of V.

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6. State the definition of each of the followings:

a. Subspace of a vector space

b. A linearly independent set of vectors

c. A spanning set of a vector space V

7. Given $A = \begin{bmatrix} 3 & 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -12 \end{bmatrix}$.

a. Find a basis for row space of A

b. Find a basis for column space of A

c. Find a basis for $N(A)$

d. Find the Rank of A

e. Find the nullity of A

8. Find a basis for $span\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -9 \\ -10 \end{bmatrix} \right\}$. Given: $\begin{bmatrix} 1 & 0 & 2 \\ 3 & -6 & -6 \\ -2 & 3 & 2 \\ 4 & -9 & -10 \end{bmatrix} \xrightarrow{RR} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

9. Prove $S = \{2 + x + x^2, x - x^2, 6 + x + 5x^2\}$ is or is not a basis of P_2.

10. Given $S = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$. If possible, write $\vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ as a linear combination of vectors in S.

11. (Extra credit) Given $A_{n\times n}, B_{n\times n}$. Prove that if $BA\vec{x} = \vec{x}, \forall \vec{x} \in \mathbb{R}^n$, then A and B are inverses of each other.

(1) V a v.s $\Rightarrow 0 \odot \hat{v} = \hat{0}$, $\forall \hat{v} \in V$.

Proof: Given V is a v.s., let $\hat{v} \in V$

$$0 + 0 = 0$$

$$(0+0)\odot\hat{v} = 0\odot\hat{v} \qquad \text{closure under } \odot$$

(10)

$$0\odot\hat{v} \oplus 0\odot\hat{v} = 0\odot\hat{v} \qquad \text{dist. prop.}$$

$$\underbrace{-(0\odot\hat{v}) \oplus 0\odot\hat{v}}_{\hat{0}} \oplus 0\odot\hat{v} = \underbrace{-(0\odot\hat{v}) \oplus 0\odot\hat{v}}_{\hat{0}}, \quad \exists \text{ of add. inverse.}$$

$$\underbrace{\hat{0} \oplus 0\odot\hat{v}}= \hat{0} \qquad \text{, prop. of " "}$$

prop. of add. identity

$\therefore$ $\boxed{0\odot\hat{v} = \hat{0}}$ QED.

(2) $S = \{\hat{v}_1, \dots, \hat{v}_k\} \subseteq V$ a v.s, At least one vector in S is a lin. comb. The rest $\Rightarrow$ S is lin. dep.

Proof: Given $S = \{\hat{v}_1, \dots, \hat{v}_k\} \subseteq V$ a v.s.

(8)

W.L.O.G, let $\hat{v}_1 = \alpha_2\hat{v}_2 + \dots + \alpha_k\hat{v}_k$

$\Rightarrow -1\hat{v}_1 + \alpha_2\hat{v}_2 + \dots + \alpha_k\hat{v}_k = \hat{0}$, coeff. of $\hat{v}_1$ is non-zero

$\therefore$ $\boxed{S \text{ is lin. dep.}}$ QED

(3) $B = \{\hat{v}_1, \dots, \hat{v}_n\}$ is a Basis of V.S. V. $S = \{\hat{u}_1, \dots, \hat{u}_m\} \subseteq V$. $m > n \Rightarrow S$ is lin. dep.

Proof: Given $B = \{\hat{v}_1, \dots, \hat{v}_n\}$ is a Basis of V, $S = \{\hat{u}_1, \dots, \hat{u}_m\} \subseteq V$ & $m > n$

Consider $\alpha_1\hat{u}_1 + \dots + \alpha_m\hat{u}_m = \hat{0}$ (1) find α_i's.

(12)

$$\begin{array}{l} \hat{u}_1 \in V \Rightarrow \\ \vdots \\ \hat{u}_m \in V \Rightarrow \end{array} \begin{cases} \hat{u}_1 = c_{11}\hat{v}_1 + \dots + c_{1n}\hat{v}_n \\ \vdots \\ \hat{u}_m = c_{m1}\hat{v}_1 + \dots + c_{mn}\hat{v}_n \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1\hat{u}_1 = \alpha_1 c_{11}\hat{v}_1 + \dots + \alpha_1 c_{1n}\hat{v}_n \\ \vdots \\ \alpha_m\hat{u}_m = \alpha_m c_{m1}\hat{v}_1 + \dots + \alpha_m c_{mn}\hat{v}_n \end{cases}$$

$+$ $\{$ by (1)

$$\hat{0} = (\alpha_1 c_{11} + \dots + \alpha_m c_{m1})\hat{v}_1 + \dots + (\alpha_1 c_{1n} + \dots + \alpha_m c_{mn})\hat{v}_n$$

$B = \{\hat{v}_1, \dots, \hat{v}_n\}$ is a Basis $\Rightarrow$ B is lin. indep.

$$\Rightarrow \left.\begin{cases} \alpha_1 c_{11} + \dots + \alpha_m c_{m1} = 0 \\ \vdots \\ \alpha_1 c_{1n} + \dots + \alpha_m c_{mn} = 0 \end{cases}\right\} \text{ONLY} \quad (2)$$

$\left.\begin{array}{l}\text{sys of eq. (2) has } m \text{ unknown \& } n\text{-eq.} \\ m > n\end{array}\right\} \Rightarrow$ (2) has ∞-many sol'ns for α_i's

$\Rightarrow$ eq (1) has ∞-many sol'ns for α_i's

$\therefore$ $\boxed{S \text{ is lin. dep.}}$ QED

(4) $A_{m\times n} \Rightarrow N(A)$ is a subsp. of $\mathbb{R}^n$.

Proof: Given $A_{m\times n}$, $N(A) = \{\vec{x} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{0}_{m\times 1}\}$

(1) $N(A) \subseteq \mathbb{R}^n$ (by Def.)

(2) $\left.\begin{array}{l}\vec{0} \in \mathbb{R}^n \\ A\vec{0}_{n\times 1} = \vec{0}_{m\times 1}\end{array}\right\} \Rightarrow \vec{0} \in N(A) \Rightarrow \boxed{N(A) \neq \emptyset}$

(3) Closure under (+):

let $\vec{x}_1 \in N(A) \Rightarrow A\vec{x}_1 = \vec{0}$, $\vec{x}_1 \in \mathbb{R}^n$

" $\vec{x}_2 \in N(A) \Rightarrow A\vec{x}_2 = \vec{0}$, $\vec{x}_2 \in \mathbb{R}^n$

$A(\vec{x}_1 + \vec{x}_2) = \vec{0}$, $(\vec{x}_1 + \vec{x}_2) \in \mathbb{R}^n$ a v.s.

$\Rightarrow (\vec{x}_1 + \vec{x}_2) \in N(A)$, $\boxed{\text{hence } N(A) \text{ has closure } (+)}$

(4) closure under (·):

let $\left.\begin{array}{l}\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0} \\ \alpha \in \mathbb{R}\end{array}\right\} \Rightarrow \alpha A\vec{x} = \alpha\vec{0}$

$A(\alpha\vec{x}) = \vec{0}$

$\Rightarrow (\alpha\vec{x}) \in N(A)$, $\boxed{\text{hence } N(A) \text{ has closure } (\cdot)}$

by (1), (2), (3) & (4) $\therefore \boxed{N(A) \text{ is a subsp. of } \mathbb{R}^n}$ QED

(10)

(10)

(5) $\dim(V) = n$ (V a v.s.), $B = \{\hat{v}_1, \ldots, \hat{v}_n\}$ is lin. indep. $\Rightarrow B$ is a Basis of V.

Proof: Given $\dim(V) = n$, $B = \{\hat{v}_1, \ldots, \hat{v}_n\} \subseteq V$ & B is lin. indep. (show B spans V)

by contradiction: suppose B does not span V

$\Rightarrow \exists\, \hat{u} \in V$ s.t. $\hat{u}$ can not be written as a lin. comb. of $\hat{v}_i$'s.

let $S = \{\hat{u}, \hat{v}_1, \ldots, \hat{v}_n\} \Rightarrow S$ is lin. indep. (1)

$\dim(V) = n \Rightarrow$ a Basis of V has n vectors

S has $n+1$ vector $\Longrightarrow$ by Thm, S is lin. dep. (2)

(1) & (2) are contradictions of each other

$\Rightarrow$ our supposition was wrong.

$\left.\begin{array}{l}\Rightarrow B \text{ spans } V \\ B \text{ is lin. indep.}\end{array}\right\} \Rightarrow$

$\therefore \boxed{B \text{ is a Basis of } V}$ QED

(6) Ref. (a) subsp.: If ① V is a v.s., ② $W \subseteq V$, ③ W is a v.s. $\xLeftrightarrow{\text{Then}}$ W i.s.T.b. a subsp. of V. (5)

(b) lin. indep. set of vectors: Given V a v.s., $S = \{\hat{v}_1, \dots, \hat{v}_K\} \subseteq V$

Consider $\alpha_1 \hat{v}_1 + \cdots + \alpha_K \hat{v}_K = \hat{0}$ ①

If eq. ① has Trivial sol'n ONLY (α_i's are all zero only) (6)

$\xLeftrightarrow{\text{Then}}$ S i.s.T.b. a lin. indep. set.

(c) A spanning set of a v.s. V: Given V a v.s., $S = \{\hat{v}_1, \dots, \hat{v}_K\} \subseteq V$ (5)

If $\text{span}(S) = V \Longleftrightarrow$ S i.s.T.b. a spanning set of V.

(7) $$A = \begin{bmatrix} 3 & 0 & 1 & 2 & 1 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 1 & 0 \end{bmatrix} \xrightarrow{RR} B = \begin{bmatrix} 1 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -12 \end{bmatrix}$$

(a) $\boxed{\text{Bases of } R_A = \{[1,0,0,0,7], [0,0,1,0,4], [0,0,0,1,-12]\}}$ (2)

(b) $\boxed{\text{Basis for } C_A = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}}$ (2)

(c) Basis for N(A): $\underset{3\times5}{A}\, \underset{5\times1}{\vec{x}} = \underset{3\times1}{\vec{0}}$ let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = ?$

$$[A \mid \vec{0}] \xrightarrow{RR} [B \mid \vec{0}] \Rightarrow \begin{cases} x_1 + 7x_5 = 0 \\ x_2 = t \\ x_3 + 4x_5 = 0 \\ x_4 - 12x_5 = 0 \\ x_5 = k \end{cases} \Rightarrow \begin{cases} x_1 = 0t - 7k \\ x_2 = 1t + 0k \\ x_3 = 0t - 4k \\ x_4 = 0t + 12k \\ x_5 = 0t + 1k. \end{cases}$$ (6)

$$\Rightarrow \vec{x} = t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} -7 \\ 0 \\ -4 \\ 12 \\ 1 \end{bmatrix} \Rightarrow \boxed{\text{Basis for } N(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ -4 \\ 12 \\ 1 \end{bmatrix} \right\}}$$

(d) $\boxed{\rho(A) = \dim(R_A) = 3}$ (2)

(e) $\boxed{\nu(A) = 2}$ (2)

(8) Find Base for $\text{Span}\left\{\underbrace{\begin{bmatrix}1\\0\\2\end{bmatrix}}_{\vec{c_1}}, \underbrace{\begin{bmatrix}3\\-6\\-6\end{bmatrix}}_{\vec{c_2}}, \underbrace{\begin{bmatrix}-2\\3\\2\end{bmatrix}}_{\vec{c_3}}, \underbrace{\begin{bmatrix}4\\-9\\-10\end{bmatrix}}_{\vec{c_4}}\right\} = W$

let $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -6 & -6 \\ -2 & 3 & 2 \\ 4 & -9 & -10 \end{bmatrix} \begin{matrix} \to \vec{c_1}^T \\ \to \vec{c_2}^T \\ \to \vec{c_3}^T \\ \to \vec{c_4}^T \end{matrix} \xrightarrow{RR} B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(6)

Basis for $R_A = \{[1,0,2], [0,1,2]\}$

$\Rightarrow \boxed{\text{Basis for } W = \left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}}$

(9) Prove $S = \{2+x+x^2,\ x-x^2,\ 6+x+5x^2\}$ is / is NOT a Basis of P_2.

Proof: Given $\dim(P_2) = 3 \Rightarrow S$ has enough vectors, let's check lin. dep./indep.

consider $\alpha_1(2+x+x^2) + \alpha_2(x-x^2) + \alpha_3(6+x+5x^2) \equiv 0$ ①, $\alpha_i = ?$

$(2\alpha_1 + 6\alpha_3) + (\alpha_1 + \alpha_2 + \alpha_3)x + (\alpha_1 - \alpha_2 + 5\alpha_3)x^2 \equiv 0, \ \forall x$

(8)

$\Rightarrow \begin{cases} 2\alpha_1 + 6\alpha_3 = 0 \\ \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ \alpha_1 - \alpha_2 + 5\alpha_3 = 0 \end{cases}$ ② $\Rightarrow A = \begin{bmatrix} 2 & 0 & 6 \\ 1 & 1 & 1 \\ 1 & -1 & 5 \end{bmatrix}$

$|A| = 2\begin{vmatrix} 1 & 1 \\ -1 & 5 \end{vmatrix} + 6\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \Rightarrow |A| = 2(6) + 6(-2) \Rightarrow |A| = 0$

$\Rightarrow$ homog. sys. of eq ② has ∞-many sol'ns $\Rightarrow$ eq ① has ∞-many sol'ns.

$\Rightarrow S$ is lin. dep.

$\therefore \boxed{S \text{ is NOT a Basis of } P_2}$ QED

(10) $S=\left\{\begin{bmatrix}1\\3\end{bmatrix},\begin{bmatrix}-1\\3\end{bmatrix}\right\}$, write $\vec{v}=\begin{bmatrix}1\\-2\end{bmatrix}$ as a lin. comb. of vectors in S.

$$\vec{v}=\begin{bmatrix}1\\-2\end{bmatrix}=\alpha_1\begin{bmatrix}1\\3\end{bmatrix}+\alpha_2\begin{bmatrix}-1\\3\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c}1 & -1 & 1\\3 & 3 & -2\end{array}\right]\xrightarrow{-3R_1+R_2}\left[\begin{array}{cc|c}1 & -1 & 1\\0 & 6 & -5\end{array}\right]\xrightarrow{\frac{1}{6}R_2}\left[\begin{array}{cc|c}1 & -1 & 1\\0 & 1 & \frac{-5}{6}\end{array}\right]\xrightarrow{R_2+R_1\to R_1}$$

(6)

$$\left[\begin{array}{cc|c}1 & 0 & \frac{1}{6}\\0 & 1 & \frac{-5}{6}\end{array}\right]\Rightarrow \left\{\alpha_1=\frac{1}{6},\ \alpha_2=\frac{-5}{6}\right.$$

$$\Rightarrow \boxed{\vec{v}=\frac{1}{6}\begin{bmatrix}1\\3\end{bmatrix}-\frac{5}{6}\begin{bmatrix}-1\\3\end{bmatrix}}$$

(Extra Credit)
(11) $A_{n\times n}$, $B_{n\times n}$, $BA\vec{x}=\vec{x}\ \forall\vec{x}\in\mathbb{R}^n \Rightarrow$ A & B are inverses of each other.

+5

Proof: Given $A_{n\times n}$, $B_{n\times n}$, $BA\vec{x}=\vec{x};\ \forall\vec{x}\in\mathbb{R}^n$

$BA\vec{x}-\vec{x}=\vec{0};$ "

$(BA-I_n)\vec{x}=\vec{0},\ \forall\vec{x}\in\mathbb{R}^n$

$\left.\begin{array}{l}\Rightarrow \vec{x}\in N(BA-I_n)\\ \ \ \& \ \forall\vec{x}\in\mathbb{R}^n\end{array}\right\}\Rightarrow N(BA-I_n)=\mathbb{R}^n$

$\left.\begin{array}{l}\Rightarrow \nu(BA-I_n)=n\\ \rho(BA-I_n)+\nu(BA-I_n)=n\end{array}\right\}\Rightarrow \rho(BA-I_n)=0$

# of pivots in $BA-I_n$

$\Rightarrow (BA-I_n)_{n\times n}$ has no pivot

$\Rightarrow BA-I_n=O_{n\times n}$

$\Rightarrow BA=I_n$

by lemma:

$\therefore \boxed{\exists A^{-1}=B\ \&\ \exists B^{-1}=A}$ Q.E.D