MATH 13 **EXAM #3** NAME: Key.

Use both sides of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your logic and ultimately your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline and mathematical rigor that was illustrated in the lecture for the proofs.

1. Prove that if $T: V \to W$ is a linear transformation then $T\left(\hat{0}_V\right) = \hat{0}_W$.
2. Prove that if V and W are isomorphic then $\dim(V) = \dim(W) = n$.
3. If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, prove that $\exists A$ such that $T(\vec{v}) = A\vec{v}$ (state the size of A and then prove).
4. Given an invertible matrix $P_{n \times n}$. If $P^{-1} = P^T$, then prove that the column (or row) vectors of P form an orthonormal set (*prove only one of them, columns or rows – your choice*).
5. Given $T: V \to W$ is a linear transformation. Prove that if $\ker(T) = \left\{\hat{0}_V\right\}$, then T is one-to-one.

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6. State the following definitions:
   - **a.** Eigenvalues and Eigenvectors of A
   - **b.** one-to-one linear transformation
   - **c.** Linear Transformation

7. Given $T: R^3 \to P_2$ such that $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 3a + (a-b)x + (2b)x^2$.
   - **a.** Prove T is a linear transformation.
   - **b.** Find the standard transformation matrix for T.
   - **c.** Find a basis for $\ker(T)$.
   - **d.** Find nullity of T.
   - **e.** Find a basis for the Range(T).
   - **f.** Find Rank of T.

8. Given $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
   - **a.** Find Eigenvalues of A.
   - **b.** Find the Eigenvectors of A corresponding to each Eigenvalue.
   - **c.** Find a basis for all Eigenspaces of A.
   - **d.** Find the Geometric multiplicity of each Eigenvalue of A.

9. (Extra Credit) Prove that if $A_{n \times n}$ is invertible, and λ_i is an eigenvalue of A, then $\lambda_i \neq 0$ and $\dfrac{1}{\lambda_i}$ is an eigenvalue of A^{-1}.

① $T: V \longrightarrow W$ a lin. Transf. $\Rightarrow T(\hat{0}_V) = \hat{0}_W$

Proof: Given $T: V \to W$ is a lin. Transf.

V is v.s $\Rightarrow \hat{0}_V \in V \Rightarrow \hat{0}_V = 0\hat{v} \quad \forall \hat{v} \in V$

$\Rightarrow T(\hat{0}_V) = T(0\hat{v})$

$= 0\,T(\hat{v})$ by pres. ⊙

$\left.\begin{array}{l} T(\hat{v}) \in W \\ W \text{ is a v.s} \end{array}\right\} \Rightarrow 0T(\hat{v}) = \hat{0}_W$

$\Rightarrow \therefore \boxed{T(\hat{0}_V) = \hat{0}_W}$ Q.E.D

(8)

② V & W are isomorphic $\Rightarrow \dim(V) = \dim(W) = n$

Proof: V & W are isomorphic V.S's., let $\dim(V) = n$

$\Rightarrow \exists\, T: V \to W$ S.T. T is an isomorphism.

$\Rightarrow T$ is 1 To 1 $\Rightarrow \nu(T) = 0$

$\left.\begin{array}{l} \nu(T) = 0 \\ \rho(T) + \nu(T) = n \end{array}\right\} \Rightarrow \rho(T) = n = \dim(V)$

T is onto $\Rightarrow \text{Range}(T) = W \Rightarrow \rho(T) = \dim(W)$

$\Rightarrow \therefore \boxed{\dim(V) = \dim(W) = n}$ Q.E.D.

(8)

③ $T: \mathbb{R}^n \to \mathbb{R}^m$ is a lin. Transf. $\Rightarrow \exists\, A_{m\times n}$ s.t. $T(\vec{v}) = A\vec{v}$

Proof: Given $T: \mathbb{R}^n \to \mathbb{R}^m$ is a lin. Transf., let $S = \{\vec{e}_1, \ldots, \vec{e}_n\}$ be The STD. Basis of $\mathbb{R}^n$

let $\vec{v} \in \mathbb{R}^n \Rightarrow \vec{v} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \Rightarrow \vec{v} = \alpha_1 \vec{e}_1 + \cdots + \alpha_n \vec{e}_n$

$\Rightarrow \boxed{T(\vec{v}) = \alpha_1 T(\vec{e}_1) + \cdots + \alpha_n T(\vec{e}_n)}$ ①

let $T(\vec{e}_1) = \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}, \ldots, T(\vec{e}_n) = \begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{mn} \end{bmatrix}$, subst. into ①

$\Rightarrow T(\vec{v}) = \alpha_1 \begin{bmatrix} c_{11} \\ \vdots \\ c_{m1} \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} c_{1n} \\ c_{2n} \\ \vdots \\ c_{mn} \end{bmatrix}$

(12)

$T(\vec{v}) = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & & c_{mn} \end{bmatrix}}_{A_{m\times n}} \underbrace{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\vec{v}}$

$T(\vec{v}) = A_{m\times n} \vec{v}$

$\therefore \boxed{\exists\, A_{m\times n} \text{ S.T. } T(\vec{v}) = A_{m\times n}\vec{v}}$ Q.E.D

(4) $P_{n\times n}$, $\exists P^{-1}$. If $P^{-1} = P^T \Rightarrow$ Col'n vectors of P form an orthonormal set.

Proof: Given $P_{n\times n}$, $\exists P^{-1} = P^T \Rightarrow \boxed{P^T P = I_n}$ (1)

let $P = [\vec{c}_1, \dots, \vec{c}_n]$ where $\vec{c}_i \in \mathbb{R}^n$ (Col'n vectors of P)

$\Rightarrow P^T = \begin{bmatrix} \vec{c}_1^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix}$, (Note $\vec{c}_i^{\,T}$ are row vectors now)

Subst. into (1):

$$\begin{bmatrix} \vec{c}_1^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix} [\vec{c}_1, \dots, \vec{c}_n] = I_n$$

$$\Rightarrow \begin{bmatrix} \vec{c}_1^{\,T}\vec{c}_1 & \vec{c}_1^{\,T}\vec{c}_2 & \cdots & \vec{c}_1^{\,T}\vec{c}_n \\ \vdots & & & \vdots \\ \vec{c}_n^{\,T}\vec{c}_1 & \vec{c}_n^{\,T}\vec{c}_2 & \cdots & \vec{c}_n^{\,T}\vec{c}_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$\begin{bmatrix} \vec{c}_1\cdot\vec{c}_1 & \vec{c}_1\cdot\vec{c}_2 & \cdots & \vec{c}_1\cdot\vec{c}_n \\ \vdots & & & \\ \vec{c}_n\cdot\vec{c}_1 & \vec{c}_n\cdot\vec{c}_2 & \cdots & \vec{c}_n\cdot\vec{c}_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \vec{c}_i \cdot \vec{c}_j = 0 & \text{when } i \neq j \\ \vec{c}_i \cdot \vec{c}_j = 1 & \text{when } i = j \end{cases}$$

$\therefore \boxed{\{\vec{c}_1, \dots, \vec{c}_n\} \text{ is an orthonormal set}}$ Q.E.D

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(5) T is a lin. Transf. If $\mathrm{Ker}(T) = \{\hat{0}_V\} \Rightarrow T$ is 1To1.

Proof: Given $T: V \to W$ is a lin. Transf. & $\mathrm{Ker}(T) = \{\hat{0}_V\}$ (show T is 1To1)

Consider $\boxed{T(\hat{v}_1) = T(\hat{v}_2)}$ for $\hat{v}_1, \hat{v}_2 \in V$ (1)

$\Rightarrow T(\hat{v}_1) - T(\hat{v}_2) = \hat{0}_W$

$\Rightarrow T(\hat{v}_1 - \hat{v}_2) = \hat{0}_W$

$$\left.\begin{array}{l} \Rightarrow (\hat{v}_1 - \hat{v}_2) \in \mathrm{Ker}(T) \\ \text{Given } \mathrm{Ker}(T) = \{\hat{0}_V\} \end{array}\right\} \Rightarrow \hat{v}_1 - \hat{v}_2 = \hat{0}_V$$

$\Rightarrow \boxed{\hat{v}_1 = \hat{v}_2}$ (2)

$\Rightarrow \boxed{\text{(1) implies (2) } \forall\, \hat{v}_1, \hat{v}_2 \in V}$

$\therefore \boxed{T \text{ is 1To1}}$ Q.E.D

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⑥ **Def:** ⓐ **eigenvalues & eigenvectors of A:** (4)

Given $A_{n\times n}$, If $\exists\ \vec{v}\in\mathbb{R}^n,\ \vec{v}\neq\vec{0}$ s.t. $A\vec{v}=\lambda\vec{v}$, $\lambda\in\mathbb{R}$

Then λ i.s.t.b. an eigenvalue of A & $\vec{v}$ i.s.t.b. an eigenvector of A Corresp. To λ.

ⓑ **1 to 1 lin. Transf.:** Given $T: V\to W$ a lin. Transf. (4)

If "$T(\hat{v}_1)=T(\hat{v}_2)$ implies $\hat{v}_1=\hat{v}_2$" $\forall\ \hat{v}_1,\hat{v}_2\in V$ Then T i.s.t.b. 1 To 1

ⓒ **lin. Transf.:**

Given $T: V\to W$ s.t. $T(\hat{v})=\hat{w}$ a function

If $\begin{cases} ① \ T(\hat{v}_1+\hat{v}_2)=T(\hat{v}_1)+T(\hat{v}_2) & (\text{pres.}\oplus) \\ ② \ T(\alpha\hat{v}_1)=\alpha T(\hat{v}_1) & (\text{Pres.}\odot)\end{cases}$ (4)

Then T i.s.t.b. a lin. Transf.

⑦ $T:\mathbb{R}^3\to P_2$ s.t. $T\begin{bmatrix}a\\b\\c\end{bmatrix}=3a+(a-b)x+(2b)x^2$

ⓐ **prove T is a lin. Transf.:**

proof: Given $T:\mathbb{R}^3\to P_2$ s.t. $T\begin{bmatrix}a\\b\\c\end{bmatrix}=3a+(a-b)x+(2b)x^2$ (9)

① show pres. $\oplus$: let $\vec{v}_1=\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}\in\mathbb{R}^3 \Rightarrow T\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}=3a_1+(a_1-b_1)x+(2b_1)x^2$

let $\vec{v}_2=\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}\in\mathbb{R}^3 \Rightarrow T\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}=3a_2+(a_2-b_2)x+(2b_2)x^2$

$\mathbb{R}^3$ is a v.s. $\Rightarrow \vec{v}_1+\vec{v}_2=\begin{bmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\end{bmatrix}\in\mathbb{R}^3$, $T\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix}+T\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix}=3(a_1+a_2)+\left[(a_1+a_2)-(b_1+b_2)\right]x+\left[2(b_1+b_2)\right]x^2$

$\Downarrow$

$T(\vec{v}_1+\vec{v}_2)=3(a_1+a_2)+\left[(a_1+a_2)-(b_1+b_2)\right]x+\left[2(b_1+b_2)\right]x^2$

$\Rightarrow \boxed{T(\vec{v}_1+\vec{v}_2)=T(\vec{v}_1)+T(\vec{v}_2)}$ hence T pres. $\oplus$.

② Show pres. $\odot$:

$\vec{v}=\begin{bmatrix}a\\b\\c\end{bmatrix}\in\mathbb{R}^3 \Rightarrow T(\vec{v})=3a+(a-b)x+2bx^2$

$\alpha\in\mathbb{R}\Rightarrow \alpha T(\vec{v})=3(\alpha a)+(\alpha a-\alpha b)x+2(\alpha b)x^2$

$\alpha\vec{v}=\begin{bmatrix}\alpha a\\\alpha b\\\alpha c\end{bmatrix}\in\mathbb{R}^3\Rightarrow T(\alpha\vec{v})=3(\alpha a)+(\alpha a-\alpha b)x+2(\alpha b)x^2$

$\Rightarrow \boxed{T(\alpha\vec{v})=\alpha T(\vec{v})}$

hence T pres. $\odot$

by ① & ②:

$\therefore$ $\boxed{\text{T is a lin. Transf.}}$ Q.E.D.

7-continued

7(b) STD Transf. matrix for T:

$S = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ is STD. Basis of $\mathbb{R}^3$

$S' = \{1, x, x^2\}$ is STD. Basis of P_2

$$T\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\vec{e}_1} = 3 + 1x + 0x^2 \Rightarrow \left[T(\vec{e}_1)\right]_{S'} = \begin{bmatrix}3\\1\\0\end{bmatrix}$$

$$T\underbrace{\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\vec{e}_2} = 0 + (-1)x + 2x^2 \Rightarrow \left[T(\vec{e}_2)\right]_{S'} = \begin{bmatrix}0\\-1\\2\end{bmatrix}$$

$$T\underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\vec{e}_3} = 0 + 0x + 0x^2 \Rightarrow \left[T(\vec{e}_3)\right]_{S'} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow$ STD. Transf. matrix

$$\boxed{A = \begin{bmatrix}3 & 0 & 0\\1 & -1 & 0\\0 & 2 & 0\end{bmatrix}}$$

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(c) Basis for Ker(T): let $\vec{v} \in \mathrm{Ker}(T) \Rightarrow \vec{v} = \begin{bmatrix}a\\b\\c\end{bmatrix}$ & $T(\vec{v}) = \hat{0}_{P_2}$

$\Rightarrow 3a + (a-b)x + (2b)x^2 = 0$ ← $\hat{0}_{P_2}$

$\Rightarrow \begin{cases} 3a = 0 \Rightarrow \boxed{a = 0} \\ a - b = 0 \\ 2b = 0 \Rightarrow \boxed{b = 0} \end{cases}$ but $c \in \mathbb{R}$ ← any Real #

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$\Rightarrow \vec{v} = \begin{bmatrix}0\\0\\c\end{bmatrix} \Rightarrow \boxed{\vec{v} = c\begin{bmatrix}0\\0\\1\end{bmatrix}}, c \in \mathbb{R} \Rightarrow \mathrm{Ker}(T) = \mathrm{Span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$\Rightarrow \boxed{\text{Basis Ker}(T) = \left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}}$

1

(d) $\boxed{\nu(T) = 1}$

(e) Basis for Range(T): Range$(T) = \{\hat{w} \in P_2$ s.t. $\hat{w} = 3a + (a-b)x + (2b)x^2\}$

$\Rightarrow \hat{w} = \underline{3a} + \underline{ax} - bx + 2bx^2$

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$\hat{w} = a(3+x) + b(-x+2x^2)$

$\Rightarrow$ Range$(T) = \mathrm{Span}\{(3+x), (-x+2x^2)\}$

$(3+x)$ & $(-x+2x^2)$ are NOT scalar multiples $\Rightarrow$ are lin. indep.

$\therefore \boxed{\text{Basis for Range }(T) = \{(3+x), (-x+2x^2)\}}$

(f) $\boxed{\rho(T) = 2}$

1

8) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a) $\lambda_i' = ?$ A is (upper) Triangular $\Rightarrow$ $\boxed{\lambda_1 = 2}$ (w/ Alg. Mult. of 2) & $\boxed{\lambda_2 = 1}$ (Alg. mult. of 1) (2)

b) eigenvectors:

for $\lambda_1 = 2 \Rightarrow (A - 2I_3)\vec{V} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow[-1R_3+R_1]{-1R_3} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ free

$\vec{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \end{cases} \Rightarrow$ $\boxed{\text{eigenvectors Corresp. To } \lambda_1 = 2: \quad \vec{V} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, t \neq 0}$ (3)

3) for $\lambda_2 = 1$: $(A - 1I_3)\vec{V} = \vec{0} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ free (3)

eigenvector $\vec{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -x_3 \\ x_2 = 0 \\ x_3 = t \end{cases} \Rightarrow$ $\boxed{\vec{V} = t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, t \neq 0}$

c) Basis $E_{\lambda_i} = ?$

$\boxed{\text{Basis for } E_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$ (2)

$\boxed{\text{Basis " } E_{\lambda_2} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$

d) Find Geom. Mult. of each λ_i E_{λ_i}:

$\boxed{\text{Geom. Mult. } \lambda_1 = 1}$, $\boxed{\text{Geom. Mult. } \lambda_2 = 1}$ (2)

9) Given $A_{n\times n}$, $\exists A^{-1}$, λ_i is eigenvalue of $A \Rightarrow \lambda_i \neq 0$ & $\frac{1}{\lambda_i}$ is an eigenvalue of A^{-1}

Proof: Given $\exists A^{-1}$, λ_i is an eigenvalue of A.

$\Rightarrow \quad P(\lambda) = |A - \lambda I_n| \Rightarrow \left.\begin{array}{l} P(0) = |A| \\ \exists A^{-1} \Rightarrow |A| \neq 0 \end{array}\right\} \Rightarrow P(0) \neq 0$ +5

$\Rightarrow \lambda_i = 0$ is NOT (can Not be) an eigenvalue of A $\Rightarrow$ $\boxed{\lambda_i \neq 0}$ (1)

λ_i is eigenvalue $\Rightarrow |A - \lambda_i I_n| = 0 \Rightarrow |A - \lambda_i A A^{-1}| = 0$

$\Rightarrow |A(I_n - \lambda_i A^{-1})| = 0$

$\Rightarrow |A| \left| \lambda_i \left(\frac{1}{\lambda_i} I_n - A^{-1}\right) \right| = 0$ (by (1))

$\Rightarrow |A| (\lambda_i)^n \left| -1 \left(A^{-1} - \frac{1}{\lambda_i} I_n\right) \right| = 0$

$\Rightarrow \left.\begin{array}{l} |A| (\lambda_i)^n (-1)^n \left| A^{-1} - \frac{1}{\lambda_i} I_n \right| = 0 \\ |A| \neq 0, (\lambda_i) \neq 0, (-1)^n \neq 0 \end{array}\right\} \Rightarrow \left| A^{-1} - \frac{1}{\lambda_i} I_n \right| = 0$

$\Rightarrow$ $\boxed{\frac{1}{\lambda_i} \text{ is an eigenvalue of } A^{-1}, \lambda_i \neq 0}$