MATH 2 **EXAM #4** **NAME:** Key.

1. Find the exact value of each of the followings:

a. $\tan(75°)$ **b.** $\sin(165°)$ **c.** $\cos\frac{11\pi}{12}$

2. Two angles α and β are in the same quadrant, $\sin\alpha = -\frac{4}{5}$ and $\tan\beta = \frac{1}{5}$. Find:

a. $\sin(\alpha+\beta)$ **b.** $\cos(\alpha+\beta)$ **d.** the quadrant containing $\alpha+\beta$.

3. Rewrite $\cos(3x)$ in terms of a single angle x.

4. Rewrite $\cos^4 x$ without any exponents.

5. Find all the solutions: $\sin 4x - \tan 2x = 0$

6. Verify each of the following identities by transforming one side to the next:

a. $\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$ **b.** $\frac{\cos 2\alpha}{1+\sin 2\alpha} = \frac{\cot\alpha - 1}{\cot\alpha + 1}$

7. Evaluate each of the followings:

a. $\sin\left(\arccos\frac{-3}{5}\right)$ **b.** $\sin\left(2\arcsin\frac{-1}{4}\right)$

8. Sketch the graph of each of the following trigonometric functions:

a. $y = 3\tan\left(-2x - \frac{\pi}{3}\right)$ **b.** $y = \sec\left(2x + \frac{\pi}{3}\right)$

9. An airplane that is flying horizontally, passes from a point exactly above your location. You notice that at some point the angle of elevation of the plane is 29°. After 5 seconds, the angle elevation of the plane is 25°. If the plane is traveling at speed of 100 ft per second, what is the altitude of the plane?

10. (EXTRA CREDIT) Express the following expression in terms of a single cosine term. That is:

$$a\cos x - b\sin x = (?)\cos(?)$$

① ⓐ $\tan(75^\circ) = \tan(45^\circ + 30^\circ)$

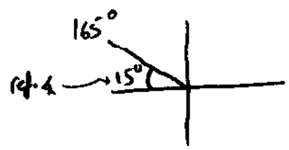
$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{\sqrt{3}}{3}}{1 - (1)\frac{\sqrt{3}}{3}}$$

$$\boxed{\tan(75^\circ) = \frac{3 + \sqrt{3}}{3 - \sqrt{3}}}$$

ⓑ $\sin(165^\circ) = ?$

(There are more than one way for this problem)



$\sin(165^\circ) = \sin(15^\circ)$

$$= \sin(60^\circ - 45^\circ)$$

$$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$$

$$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2}$$

$$\boxed{\sin(165^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}}$$

ⓒ $\cos\frac{11\pi}{12} = \cos\left(\frac{1}{2}\cdot\frac{11\pi}{6}\right)$; $\frac{11\pi}{12}$ is in Q II $\Rightarrow$ $\cos\frac{11\pi}{12}$ will be negative

$$= -\sqrt{\frac{1 + \cos\frac{11\pi}{6}}{2}}$$

$\frac{11\pi}{6} \to$ Q IV $\Rightarrow$ $\cos\frac{11\pi}{6}$ is positive.

$$= -\sqrt{\frac{1 + \frac{\sqrt{3}}{2}}{2}}$$

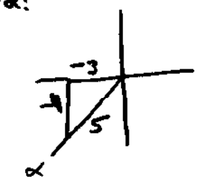
$$= -\sqrt{\frac{2 + \sqrt{3}}{4}}$$

$$\boxed{\cos\frac{11\pi}{12} = -\frac{\sqrt{2+\sqrt{3}}}{2}}$$

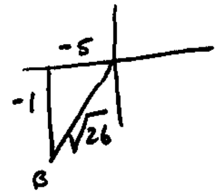
② $\sin\alpha = -\frac{4}{5}$, $\tan\beta = \frac{1}{5}$, α & β are in same Quad.

Q III or Q IV Q I or Q III $\longrightarrow$ α & β are in Q III

For α:



For β:



ⓐ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{-4}{5}\,\frac{-5}{\sqrt{26}} + \frac{-3}{5}\,\frac{-1}{\sqrt{26}}$$

$$\boxed{\sin(\alpha+\beta) = \frac{20+3}{5\sqrt{26}} = \frac{23}{5\sqrt{26}}}$$

ⓑ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \frac{-3}{5}\,\frac{-5}{\sqrt{26}} - \frac{-4}{5}\,\frac{-1}{\sqrt{26}}$$

$$= \frac{15-4}{5\sqrt{26}}$$

$$\boxed{\cos(\alpha+\beta) = \frac{11}{5\sqrt{26}}}$$

ⓒ Since $\sin(\alpha+\beta)$ & $\cos(\alpha+\beta)$ are both positive $\Rightarrow$ $\boxed{(\alpha+\beta) \text{ is in 1st Quad.}}$

③ $\cos(3x) = \cos(2x+x)$

$$= \cos 2x\cos x - \sin 2x \sin x$$

$$= (2\cos^2 x - 1)\cos x - 2\sin x\cos x \sin x$$

$$= 2\cos^3 x - \cos x - 2\sin^2 x\cos x$$ ← you could stop here too!

$$= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$\boxed{\cos(3x) = 4\cos^3 x - 3\cos x}$$

④ $\cos^4 x = (\cos^2 x)^2$

$$= \left[\tfrac{1}{2}(\cos 2x + 1)\right]^2$$

$$= \tfrac{1}{4}\left[\cos^2 2x + 2\cos 2x + 1\right]$$

$$= \tfrac{1}{4}\cos^2 2x + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}$$

$$= \tfrac{1}{4}\left[\tfrac{1}{2}(\cos 4x + 1)\right] + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}$$

$$= \tfrac{1}{8}\cos 4x + \tfrac{1}{8} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}$$

$$\boxed{\cos^4 x = \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8}}$$

⑤ $\sin 4x - \tan 2x = 0$

$$2\sin 2x \cos 2x - \tan 2x = 0$$

$$2\sin 2x \cos 2x - \frac{\sin 2x}{\cos 2x} = 0$$

$$\frac{2\sin 2x \cos^2 2x - \sin 2x}{\cos 2x} = 0 \quad ; \quad \cos 2x \neq 0$$

$$\sin 2x\left(2\cos^2 2x - 1\right) = 0$$

$\sin 2x = 0$ OR $\cos^2 2x = \frac{1}{2}$

$2x = n\pi$

$\boxed{x = \frac{n\pi}{2}}$

$\cos 2x = \pm\frac{\sqrt{2}}{2}$

$\cos 2x = \frac{\sqrt{2}}{2}$ OR $\cos 2x = -\frac{\sqrt{2}}{2}$

$2x = \frac{\pi}{4} + 2\pi n$

$\boxed{x = \frac{\pi}{8} + \pi n}$

$2x = \frac{7\pi}{4} + 2\pi n$

$\boxed{x = \frac{7\pi}{8} + \pi n}$

$2x = \frac{3\pi}{4} + 2\pi n$

$\boxed{x = \frac{3\pi}{8} + \pi n}$

$2x = \frac{5\pi}{4} + 2\pi n$

$\boxed{x = \frac{5\pi}{8} + \pi n}$

(6) (a) $RHS = (\sec\theta - \tan\theta)^2$

$$= \left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)^2$$

$$= \frac{(1-\sin\theta)^2}{\cos^2\theta}$$

$$= \frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$$= \frac{(1-\sin\theta)^{\cancel{2}}}{\cancel{(1-\sin\theta)}(1+\sin\theta)}$$

$$RHS = \frac{1-\sin\theta}{1+\sin\theta} \qquad \text{Q.E.D.}$$

(b) $$\frac{\cos 2\alpha}{1+\sin 2\alpha} = \frac{2\cos^2\alpha - 1}{1+2\sin\alpha\cos\alpha}$$

divide every Term in The num. & den. by $\sin^2\alpha$

$$= \frac{\frac{2\cos^2\alpha}{\sin^2\alpha} - \frac{1}{\sin^2\alpha}}{\frac{1}{\sin^2\alpha} + \frac{2\sin\alpha\cos\alpha}{\sin^2\alpha}}$$

$$= \frac{2\cot^2\alpha - \csc^2\alpha}{\csc^2\alpha + 2\cot\alpha}$$

want every Thing in Terms of $\cot\alpha$
$\csc^2\alpha = \cot^2\alpha + 1$

$$= \frac{2\cot^2\alpha - (\cot^2\alpha + 1)}{\cot^2\alpha + 1 + 2\cot\alpha}$$

$$= \frac{\cot^2\alpha - 1}{\cot^2\alpha + 2\cot\alpha + 1}$$

$$= \frac{\cancel{(\cot\alpha+1)}(\cot\alpha - 1)}{(\cot\alpha+1)^{\cancel{2}}}$$

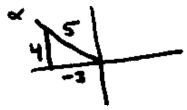
$$\boxed{\frac{\cos 2\alpha}{1+\sin 2\alpha} = \frac{\cot\alpha - 1}{\cot\alpha + 1}}$$

7 (a) $\sin\left(\underbrace{\arccos \frac{-3}{5}}_{\alpha}\right)$

let $\arccos \frac{-3}{5} = \alpha \implies$ Find $\sin\alpha = ?$

$\Downarrow$ $0 \leq \alpha \leq \pi$

$\cos\alpha = \frac{-3}{5} \implies \alpha$ in Q II



$\sin\alpha = \frac{4}{5}$

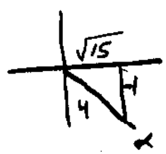
$\therefore \boxed{\sin\left(\arccos \frac{-3}{5}\right) = \frac{4}{5}}$

(b) $\sin\left(2\, \underbrace{\arcsin \frac{-1}{4}}_{\alpha}\right)$

let $\alpha = \arcsin \frac{-1}{4} \implies$ Find $\sin(2\alpha) = ?$

$\Downarrow$ $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$

$\sin\alpha = -\frac{1}{4} \implies \alpha$ in Q IV



$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\left(-\frac{1}{4}\right)\left(\frac{\sqrt{15}}{4}\right)$

$\boxed{\sin 2\alpha = \frac{-\sqrt{15}}{8}}$

$\therefore \boxed{\sin\left(2\arcsin \frac{-1}{4}\right) = \frac{-\sqrt{15}}{8}}$

(8) (a) $y = 3\tan\left(-2x - \frac{\pi}{3}\right)$

$\Rightarrow y = -3\tan\left(2x + \frac{\pi}{3}\right)$

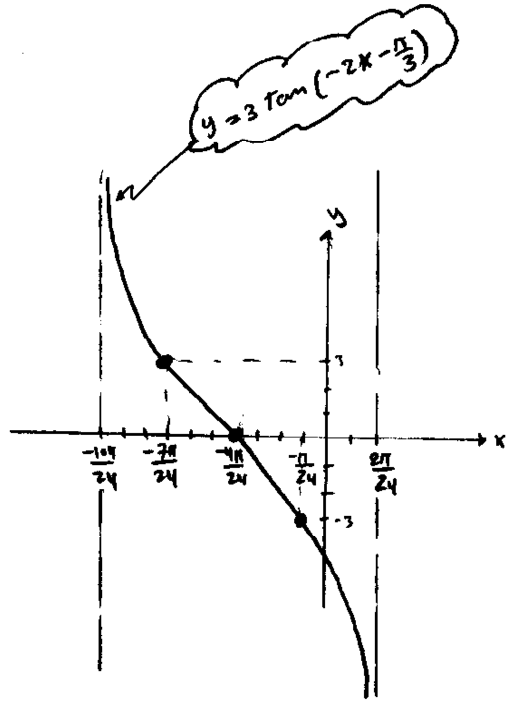
one cycle $-\frac{\pi}{2} < 2x + \frac{\pi}{3} < \frac{\pi}{2}$

$-\frac{5\pi}{6} < 2x < \frac{\pi}{6}$

$-\frac{5\pi}{12} < x < \frac{\pi}{12}$

$-\frac{5\pi}{12}$	$\frac{-7\pi}{24}$	$-\frac{3\pi}{12}$	$\frac{-\pi}{24}$	$\frac{1\pi}{12}$

$-\frac{10\pi}{24}$	$\frac{-7\pi}{24}$	$\frac{-4\pi}{24}$	$\frac{-\pi}{24}$	$\frac{2\pi}{24}$
VA	+3	on	-3	VA



(b) $y = \sec\left(2x + \frac{\pi}{3}\right)$

Graph $y = \cos\left(2x + \frac{\pi}{3}\right)$ first.

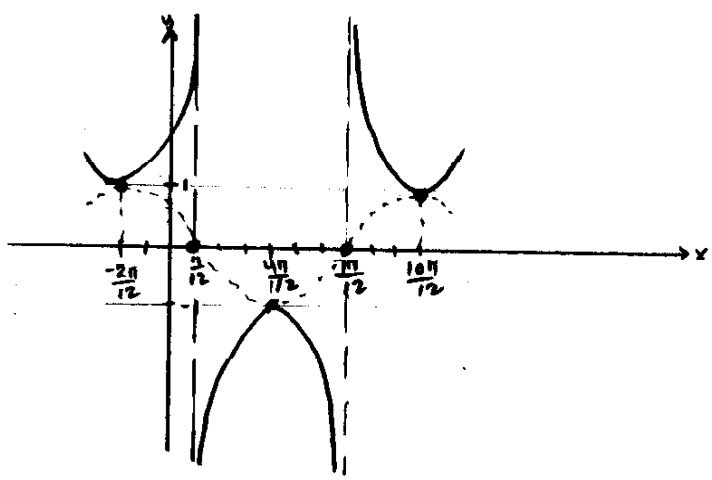
$0 \leq 2x + \frac{\pi}{3} \leq 2\pi$

$-\frac{\pi}{3} \leq 2x \leq \frac{5\pi}{3}$

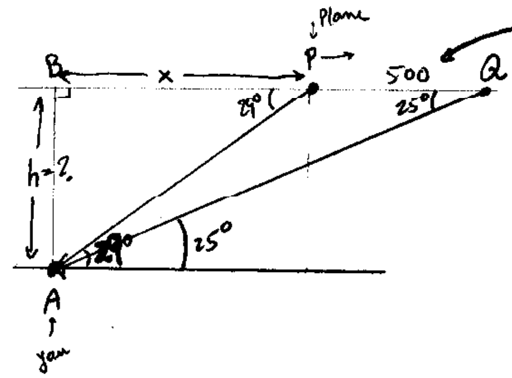
$-\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

$\frac{-\pi}{6}$	$\frac{+\pi}{12}$	$\frac{2\pi}{6}$	$\frac{7\pi}{12}$	$\frac{5\pi}{6}$

$\frac{-2\pi}{12}$	$\frac{+\pi}{12}$	$\frac{4\pi}{12}$	$\frac{7\pi}{12}$	$\frac{10\pi}{12}$
mx	on	min	on	mx



(9)



$$\left(100\ \frac{ft}{sec}\right)(5\ sec) = 500\ ft$$

in $\stackrel{\Delta}{ABP}$: $\tan 29° = \frac{h}{x} \Rightarrow x = \frac{h}{\tan 29°} = h\cot 29°$

in $\stackrel{\Delta}{ABQ}$: $\tan 25° = \frac{h}{x+500}$

$$\Rightarrow \tan 25° = \frac{h}{h\cot 29° + 500}$$

$$\tan 25°\left(h\cot 29° + 500\right) = h$$

$$h\tan 25°\cot 29° + 500\tan 25° = h$$

$$500\tan 25° = h - h\tan 25°\cot 29°$$

$$500\tan 25° = h\left(1 - \tan 25°\cot 29°\right)$$

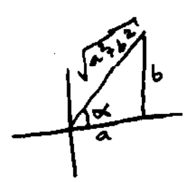
$$\Rightarrow \boxed{h = \frac{500\tan 25°}{1 - \tan 25°\cot 29°}}$$

$$\boxed{h \approx 1468.6\ ft}$$

(10) (Extra Credit)

$$a\cos x - b\sin x = (?)\cos(?)$$

consider:



so; let α be an angle whose $\tan\alpha = \frac{b}{a}$

$\Rightarrow \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ & $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

$$a\cos x - b\sin x = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos x - \frac{b}{\sqrt{a^2+b^2}}\sin x\right]$$

$$= \sqrt{a^2+b^2}\left[\cos\alpha\cos x - \sin\alpha\sin x\right]$$

$$\boxed{a\cos x - b\sin x = \sqrt{a^2+b^2}\ \cos(\alpha + x)\ ; \ \text{where } \alpha = \arctan\left(\frac{b}{a}\right)}$$