MATH 2 **EXAM #5** **NAME:** Key

1. Prove that in any triangle $\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}$.

2. Find the remaining parts of the following triangles with given information:
   a. $a=2,\ c=1,\ \gamma=25°$ **b.** $a=3,\ b=5,\ c=7$

3. Sketch the graph of each of the following equations. In each case list the coordinates of all the special points ($F_1,F_2,V_1,V_2,M_1,M_2,C$), equation of the directrix, equations of asymptotes as relevant to each graph.
   a. $y^2-4y-2x=0$ **b.** $y^2-4x^2-16x-2y-19=0$
   c. $9(x-1)^2+4(y+2)^2=36$

4. Find the equation of set of all points in the plane whose sum of distances from points $(1,2)$ and $(1,-4)$ are always 10.

5. State the geometric definition of:
   a. parabola **b.** hyperbola

6. Use matrices and row operations to solve the following system of equations.
$$\begin{cases} x-y=6 \\ 2x-3z=16 \\ 2y+z=4 \end{cases}$$

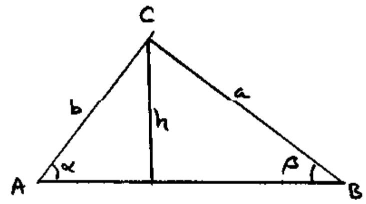
7. A student was solving a system of equations using matrices. What is the solution to the system if the final result of her operation was the following matrix. If there was no solutions, say so, otherwise write the solution in the proper form.

   a. $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

8. (EXTRA CREDIT) Use the geometric definition of a parabola to show that the general equation of a parabola with a horizontal directrix is $(x-h)^2=2p(y-k)$.

① $\sin\alpha = \frac{h}{b} \Rightarrow h = b\sin\alpha$ ①

$\sin\beta = \frac{h}{a} \Rightarrow h = a\sin\beta$ ②



① & ② $\Rightarrow$ $b\sin\alpha = a\sin\beta$

$$\frac{b\sin\alpha}{ab} = \frac{a\sin\beta}{ab}$$

$$\boxed{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}}$$

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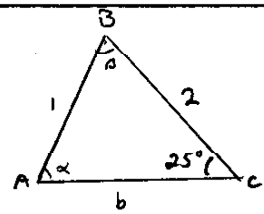
② ⓐ $\begin{cases} a = 2 \\ c = 1 \\ \gamma = 25° \end{cases} \Rightarrow \begin{cases} \alpha = ? \\ \beta = ? \\ b = ? \end{cases}$



(SSA) → watch out for a possible second solution.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$\frac{\sin\alpha}{2} = \frac{\sin\beta}{b} = \frac{\sin 25°}{1} \Rightarrow \frac{\sin\alpha}{2} = \sin 25°$$

$$\sin\alpha = 2\sin 25°$$

$$\boxed{\alpha = 57.7°}$$

$\boxed{\alpha_1 = 57.7°}$

$\beta_1 = 180 - (57.7° + 25°)$

$\beta_1 = 97.3°$

$\frac{\sin 97.3°}{b_1} = \frac{\sin 25°}{1}$

$b_1 = \frac{\sin 97.3°}{\sin 25°}$

$\boxed{b_1 = 2.3}$

$\alpha_2 = 180° - 57.7° = \boxed{122.3°} = \alpha_2$

$\beta_2 = 180° - (122.3° + 25°)$

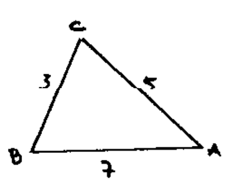
$\boxed{\beta_2 = 32.7°}$

$\frac{\sin 32.7°}{b_2} = \frac{\sin 25°}{1}$

$b_2 = \frac{\sin 32.7°}{\sin 25°}$

$\boxed{b_2 = 1.3}$

ⓑ $\begin{cases} a = 3 \\ b = 5 \\ c = 7 \end{cases}$



$a^2 = b^2 + c^2 - 2bc\cos\alpha \Rightarrow 9 = 25 + 49 - 2(5)(7)\cos\alpha$

$\cos\alpha = \frac{9-25-49}{-2(5)(7)} \Rightarrow \boxed{\alpha = 21.8°}$

$b^2 = a^2 + c^2 - 2ac\cos\beta \Rightarrow 25 = 9 + 49 - 2(3)(7)\cos\beta$

$\cos\beta = \frac{25-9-49}{-2(3)(7)} \Rightarrow \boxed{\beta = 38.2°}$

$\gamma = 180° - (21.8° + 38.2°) \Rightarrow \boxed{\gamma = 120°}$

③ ⓐ $y^2-4y-2x=0$

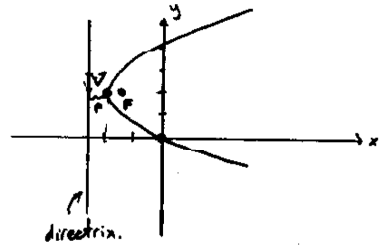
$y^2-4y+(4)=2x+(4)$

$(y-2)^2=2(x+2)$

$4p=2 \Rightarrow \boxed{p=\frac{1}{2}}$

$V(-2,2)$

if $y=0 \Rightarrow x=0 \Rightarrow (0,0)$



$\boxed{V(-2,2)}$

$\boxed{F(-1.5,2)}$

eq. of dir: $\boxed{x=-2.5}$

ⓑ $y^2-4x^2-16x-2y-19=0$

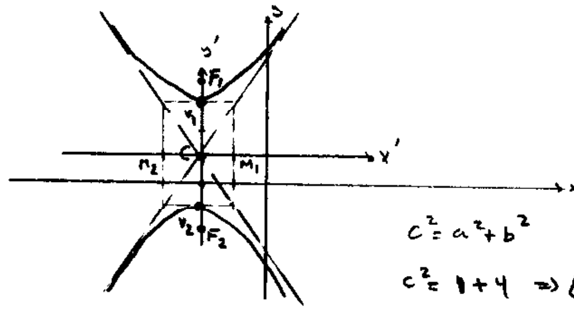
$y^2-2y-4x^2-16x=19$

$(y^2-2y+1)-4(x^2+4x+4)=19+1-16$

$(y-1)^2-4(x+2)^2=4$

$$\boxed{\frac{(y-1)^2}{4}-\frac{(x+2)^2}{1}=1}$$

$\boxed{C(-2,1)}$



$c^2=a^2+b^2$

$c^2=1+4 \Rightarrow \boxed{c=\sqrt{5}}$

$\boxed{V_1(-2,3)}$ $\boxed{M_1(-1,1)}$ $\boxed{F_1(-2,1+\sqrt{5})}$

$\boxed{V_2(-2,-1)}$ $\boxed{M_2(-3,1)}$ $\boxed{F_2(-2,1-\sqrt{5})}$

Eq. of asymptotes: $\dfrac{(y-1)^2}{4}-\dfrac{(x+2)^2}{1}=0$

$(y-1)^2=4(x+2)^2$

$y-1=\pm 2(x+2)$

$y=1\pm 2(x+2)$ → $y=1+2x+4 \Rightarrow \boxed{y=2x+5}$

→ $y=1-2x-4 \Rightarrow \boxed{y=-2x-3}$

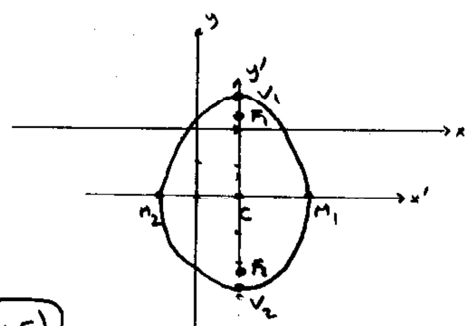
ⓒ $9(x-1)^2+4(y+2)^2=36$

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{9}=1$$

$\Rightarrow \boxed{a=3}, \boxed{b=2}$

$\boxed{C(1,-2)}$

$c^2=a^2-b^2 \Rightarrow c=9-4 \Rightarrow \boxed{c=\sqrt{5}}$

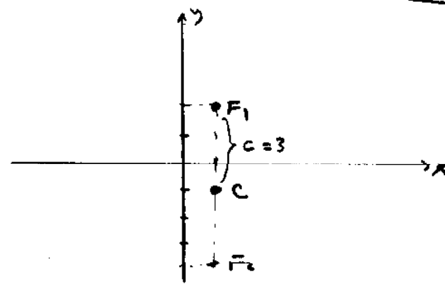


$\boxed{V_1(1,1)}$ $\boxed{M_1(3,-2)}$ $\boxed{F_1(1,-2+\sqrt{5})}$

$\boxed{V_2(1,-5)}$ $\boxed{M_2(-1,-2)}$ $\boxed{F_2(1,-2-\sqrt{5})}$

(4) it's an **ellipse**: with Foci at $(1,2)$ & $(1,-4)$

& $2a=10 \Rightarrow \boxed{a=5}$



$c(1,-1)$

$$\left.\begin{array}{l} a=5 \\ c=3 \\ c^2=a^2-b^2 \end{array}\right\} \Rightarrow 9=25-b^2 \Rightarrow b^2=25-9$$

$$b^2=16 \Rightarrow b=4$$

$$\boxed{\frac{(x-1)^2}{16}+\frac{(y+1)^2}{25}=1}$$

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(5) (a) Parabola: A parabola is a set of all points in the plane that are equidistance from a fixed point (Focus) & a line (directrix) in the same plane.

(b) Hyperbola: A hyperbola is a set of all points in the plane whose difference of distances from two fixed pts in the plane (Foci) is a constant = $2a$.

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(6) $\begin{cases} x-y=6 \\ 2x-3z=16 \\ 2y+z=4 \end{cases}$

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 6 \\ 2 & 0 & -3 & 16 \\ 0 & 2 & 1 & 4 \end{array}\right] \xrightarrow{-2R_1+R_2\to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 6 \\ 0 & 2 & -3 & 4 \\ 0 & 2 & 1 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 6 \\ 0 & \boxed{1} & -\frac{3}{2} & 2 \\ 0 & 2 & 1 & 4 \end{array}\right] \xrightarrow[-2R_2+R_3\to R_3]{R_2+R_1\to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 8 \\ 0 & 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 4 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_3}$$

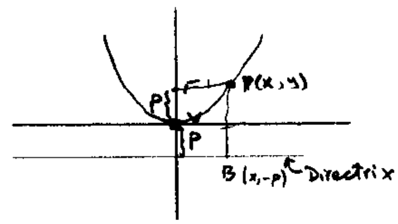
$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 8 \\ 0 & 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\frac{3}{2}R_3+R_2\to R_2]{\frac{3}{2}R_3+R_1\to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \boxed{\begin{cases} x=8 \\ y=2 \\ z=0 \end{cases}}$$

7 (a) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x+2z=1 \Rightarrow & x=-2z+1 \\ y-z=3 \Rightarrow & y=z+3 \end{cases} \xrightarrow{\text{let } z=t} \boxed{\begin{cases} x=-2t+1 \\ y=t+3 \\ z=t \end{cases}}$ infinitely many solutions.

(b) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right] \Rightarrow \begin{cases} x+2z=1 \\ y-z=3 \\ 0=-1 \longrightarrow \text{Not possible} \end{cases} \longrightarrow \therefore \boxed{\text{NO solution}}$

8 First show the eq. using a parabola whose vertex is at the origin & then apply horiz. & vertical shift to generalize it.



$V(0,0)$

$F(0,P)$

$B(x,-P)$

$$\left.\begin{array}{l} d\overline{PB} = d\overline{PF} \\ d\overline{PB} = \sqrt{(x-x)^2+(y+P)^2} \\ d\overline{PF} = \sqrt{(x-0)^2+(y-P)^2} \end{array}\right\} \longrightarrow \sqrt{(y+P)^2} = \sqrt{x^2+(y-P)^2}$$

$$(y+P)^2 = x^2+(y-P)^2$$

$$\cancel{y^2} + 2Py + \cancel{P^2} = x^2 + \cancel{y^2} - 2Py + \cancel{P^2}$$

$$\boxed{4Py = x^2} \longrightarrow \text{parabola with vertex at origin}$$

applying possible horiz. & vertical shift:

$$\boxed{4P(y-k) = (x-h)^2}$$