Math 7 **Exam #1** **Name:** Key

Use **both sides** of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline that was requested for the homework assignments. Also note that you can **only use formulas and methods that were presented** thus far **in the lecture**.

1. Given $f(x)=\begin{cases} x^2+1 & \text{if} \quad x\le -1 \\ -2x+3 & \text{if} \quad |x|<1 \\ \sqrt{x+1} & \text{if} \quad x>1 \end{cases}$

   a. Evaluate each of the followings: $f(-1), f(0), f(1)$

   b. Use the **definition of derivative** to **prove** that $f(x)$ is or is not differentiable at $x=-1$.

   c. **Prove** that $f(x)$ is or is not continuous at $x=-1$. If not continuous, state the type of discontinuity.

2. Use the formal definition of limit to **prove** $\lim\limits_{x\to -3}(x^2-4)=5$.

3. **Prove** the derivative formula for $\cos x$.

4. **Prove** the derivative formula for $\sec x$.

5. **Prove** the Quotient Rule of derivative.

6. State the formal **definition** of $\lim\limits_{x\to +\infty} f(x)=-\infty$ (show a graphical illustration too).

7. **Prove** $\lim\limits_{\theta\to 0^+}\dfrac{\theta}{\sin\theta}=1$.

8. Evaluate each of the following limits (when **<u>applicable</u>** clearly expose **C.Z.F.** or **Z.F.** as illustrated in the lecture):

   a. $\lim\limits_{x\to -2}\dfrac{x^3+8}{x^5+3x^2+20}$ **b.** $\lim\limits_{x\to -1}\dfrac{\sqrt[5]{x}+1}{x^2-1}$

   c. $\lim\limits_{x\to 3}\sqrt{9-x^2}$ **d.** $\lim\limits_{x\to -3}\dfrac{x+3}{|x^2-9|}$

   e. $\lim\limits_{x\to 0}\dfrac{\sin^3(3x)}{x^2\sin(2x)}$ **f.** $\lim\limits_{x\to -2}\dfrac{x^2-9}{\sqrt{(x+2)^2}}$

9. Use the **definition of derivative** to find derivative of $f(x)=\dfrac{1}{x^2}$.

10. Differentiate each of the followings w.r.t. its corresponding independent variable (simplify completely) (Recall that you may not use the chain rule in this exam):

   a. $f(x)=\dfrac{\sqrt[3]{x^2}}{x^2+1}$ **b.** $y=\dfrac{2\cos x+\sin x-3}{\sin x}$

11. Given $f(x)=\sqrt[3]{x}$.

   a. Find $f'(x)$.

   b. Find all points where there is a Vertical Tangent Line (VTL) and prove that there exist a VTL at each point.

12. (Extra Credit) Evaluate: $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\frac{\sqrt{2}}{2}}{x-\frac{\pi}{4}}$ (as usual, your *work* must clearly support your answer).

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① $f(x)=\begin{cases} x^2+1 & \text{if } x\le -1 \\ -2x+3 & \text{if } |x|<1 \Rightarrow -1<x<1 \\ \sqrt{x+1} & \text{if } x>1 \end{cases}$

(a) $f(-1) = x^2+1\big|_{x=-1}$

$\boxed{f(-1)=2}$

$f(0) = -2x+3\big|_{x=0}$

$\boxed{f(0)=3}$

$\boxed{f(1) \text{ is undefined}}$

(3)

(b) proof: $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$

$f'(-1) = \lim_{x\to -1} \frac{f(x)-f(-1)}{x+1}$ ⟶ $f(-1)=2$ from a

$= \lim_{x\to -1} \frac{f(x)-2}{x+1}$

(7)

side limit. $\lim_{x\to -1^+} \frac{f(x)-2}{x+1} = \lim_{x\to -1^+} \frac{-2x+3-2}{x+1}$

$(x>-1)$ $(x+1>0)$

$= \lim_{x\to -1^+} \frac{-2x+1}{x+1}$

$= +\infty$

inv. $\frac{3}{0^+} \to +\infty$

$\lim_{x\to -1^-} \frac{f(x)-2}{x+1} = \lim_{x\to -1^-} \frac{x^2+1-2}{x+1}$

$(x<-1)$

inv. $\frac{0}{0}$ indeterminate CZF $=(x+1)$

$= \lim_{x\to -1^-} \frac{x^2-1}{x+1}$

$= \lim_{x\to -1^-} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}}$

$= -2$

$\lim_{x\to -1^+} \frac{f(x)-2}{x+1} \neq \lim_{x\to -1^-} \frac{f(x)-2}{x+1}$

$\Rightarrow \lim_{x\to -1} \frac{f(x)-2}{x+1}$ DNE

$\Rightarrow f'(-1)$ DNE

$\therefore$ $\boxed{f(x) \text{ is Not Differentiable at } x=-1}$ Q.E.D

(c) proof: ① $f(-1)=2$ defined ✓

② $\lim_{x\to -1} f(x) = ?$

side limit: $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (-2x+3)$

$(x>-1)$

$= 5$

(4)

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+1)$

$(x<-1)$

$= 2$

$\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$

$\Rightarrow \lim_{x\to -1} f(x)$ DNE

$\therefore$ $\boxed{f(x) \text{ is NOT Continuous at } x=1 \text{ \& has a Jump Discontinuity}}$ Q.E.D

(2) $\lim_{x\to -3}(x^2-4)=5 \Rightarrow \begin{cases} f(x)=x^2-4 \\ a=-3 \\ L=5 \end{cases}$

Proof: $\forall$ (given) $\varepsilon>0$, $\exists$ (find) $\delta>0$ S.T.

If $0<|x-a|<\delta$ Then $|f(x)-L|<\varepsilon$

" $0<|x+3|<\delta$ Then $|x^2-4-5|<\varepsilon$

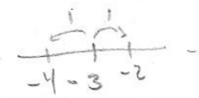
" " " $|x^2-9|<\varepsilon$

" " " $|x+3||x-3|<\varepsilon$

" " " $|x+3|<\dfrac{\varepsilon}{|x-3|}$

7

let $\delta=1 \Rightarrow$



$-4<x<-2$

$-3 \quad -3 \quad -3$

$-7<x-3<-5$

$\Rightarrow$ If $0<|x+3|<\delta$ Then $|x+3|<\dfrac{\varepsilon}{7}$

$\therefore$ $\boxed{\text{Choose } \delta\le \min\left\{1, \frac{\varepsilon}{7}\right\}}$ Q.E.D

(3) $D_x[\cos x]=-\sin x$

proof: let $f(x)=\cos x$, find $f'(x)$

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\dfrac{\cos(x+h)-\cos x}{h}$

7

$=\lim_{h\to 0}\dfrac{\cos x\cos h-\sin x\sin h-\cos x}{h}$

$=\lim_{h\to 0}\left[-\sin x\dfrac{\sin h}{h}+\cos x\dfrac{\cos h-1}{h}\right]$

$=-\sin x\underbrace{\lim_{h\to 0}\dfrac{\sin h}{h}}+\cos x\underbrace{\lim_{h\to 0}\dfrac{\cos h-1}{h}}$

$=-\sin x(1)+\cos x(0)$

$f'(x)=-\sin x$

$\therefore$ $\boxed{D_x[\cos x]=-\sin x}$ Q.E.D

(4) $D_x[\sec x]=\sec x\tan x$

Proof: let $f(x)=\sec x$, find $f'(x)$

$D_x\left(\; f(x)=\dfrac{1}{\cos x}\;\right)D_x$

$f'(x)=\dfrac{(1)'\cos x-(1)(\cos x)'}{\cos^2 x}$

7

$=\dfrac{0+\sin x}{\cos^2 x}$

$=\dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x}$

$\boxed{f'(x)=\sec x\cdot\tan x}$

$\therefore$ $\boxed{D_x[\sec x]=\sec x\tan x}$ Q.E.D

(5) $D_x\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

Proof: let $P(x) = \frac{f(x)}{g(x)}$, find $P'(x)$

$$P'(x) = \lim_{h\to 0} \frac{P(x+h) - P(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \cdot \frac{g(x+h)\,g(x)}{g(x+h)\,g(x)}$$ (8)

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x) - f(x)\,g(x+h)}{h} \cdot \frac{1}{g(x+h)\,g(x)}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - \boxed{f(x)\,g(x)} - f(x)g(x+h) + \boxed{f(x)\,g(x)}}{h} \cdot \frac{1}{g(x+h)\,g(x)}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h}\,g(x) - f(x)\,\frac{g(x+h) - g(x)}{h}\right] \cdot \lim_{h\to 0} \frac{1}{g(x+h)\,g(x)}$$

$$= \left[\left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) g(x) - f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right] \cdot \lim_{h\to 0} \frac{1}{g(x+h)\,g(x)}$$

$$= [\, f'(x)\,g(x) - f(x)\,g'(x)\,] \cdot \frac{1}{g(x)\,g(x)}$$

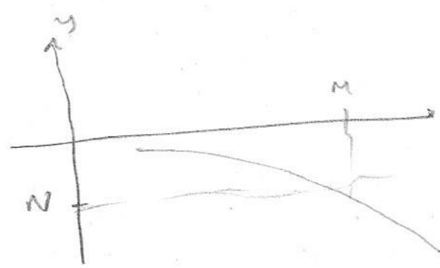
$$\therefore \boxed{D_x\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$ QED

(6) Def: $\lim_{x\to +\infty} f(x) = -\infty$ (4)

means

$\forall$ (given) $N < 0$, $\exists$ (find) $M > 0$ S.T.

If $x > M$ Then $f(x) < N$



(7) $\lim_{\theta\to 0^+} \frac{\theta}{\sin\theta} = 1$

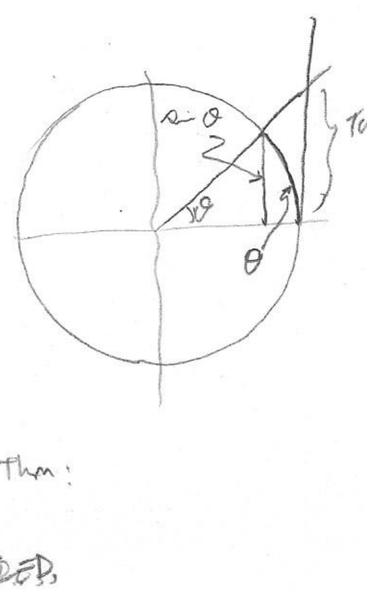
Proof: $\theta \to 0^+ \Rightarrow \theta > 0$, & $\theta \in Q\,I$

$\Rightarrow \tan\theta > 0$, $\sin\theta > 0$

from unit circle:

$\sin\theta \le \theta \le \tan\theta$

$\frac{\sin\theta}{\sin\theta} \le \frac{\theta}{\sin\theta} \le \frac{\tan\theta}{\sin\theta}$



(6)

① $1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$

② $\lim_{\theta\to 0^+} 1 = 1$ $\Longrightarrow$ by Sandwich / Squeeze Thm:

③ $\lim_{\theta\to 0^+} \frac{1}{\cos\theta} = 1$

$$\therefore \boxed{\lim_{\theta\to 0^+} \frac{\theta}{\sin\theta} = 1}$$ QED.

8 (a)

$$\lim_{x\to -2} \frac{x^3+8}{x^5+3x^2+20} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}(x^4-2x^3+4x^2-5x+10)}$$

$$= \frac{12}{68}$$

$$\boxed{\lim_{x\to -2} \frac{x^3+8}{x^5+3x^2+20} = \frac{3}{17}}$$

inv. $\frac{0}{0}$ indeterminate
CZF $= (x+2)$

$$\begin{array}{r|rrrrrr} -2 & 1 & 0 & 0 & 3 & 0 & 20 \\ & & -2 & 4 & -8 & 10 & -20 \\ \hline & 1 & -2 & 4 & -5 & 10 & \boxed{0} \end{array}$$

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(b)

$$\lim_{x\to -1} \frac{\sqrt[5]{x}+1}{x^2-1} = \lim_{x\to -1} \frac{\sqrt[5]{x}+1}{(x+1)(x-1)} \cdot \frac{c}{c}$$

$$= \lim_{x\to -1} \frac{\cancel{x+1}}{\cancel{(x+1)}(x-1)\cdot c}$$

$$= \lim_{x\to -1} \frac{1}{(x-1)\left(\sqrt[5]{x^4}-\sqrt[5]{x^3}+\sqrt[5]{x^2}-\sqrt[5]{x}+1\right)}$$

$$\boxed{\lim_{x\to -1} \frac{\sqrt[5]{x}+1}{x^2-1} = \frac{1}{-10}}$$

inv. $\frac{0}{0}$ indeterminate
CZF $= (x+1)$

have $\longrightarrow$ want

$\sqrt[5]{x}+1$ $\qquad (\sqrt[5]{x})^5+1^5 = x+1$

$(A+1)(\quad ? \quad) = A^5+1$

$$\begin{array}{r|rrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & \boxed{0} \end{array}$$

$(A+1)(A^4-A^3+A^2-A+1) = A^5+1$

$$(\sqrt[5]{x}+1)\underbrace{\left(\sqrt[5]{x^4}-\sqrt[5]{x^3}+\sqrt[5]{x^2}-\sqrt[5]{x}+1\right)}_{c} = x+1$$

5

(c)

$$\lim_{x\to 3} \sqrt{9-x^2} = \lim_{x\to 3} \sqrt{(3-x)(3+x)}$$

$$= \lim_{x\to 3} \sqrt{-(x-3)(3+x)}$$

Inv. $\sqrt{0}$ watch-out ZF $= x-3$

Side limits:

$$\lim_{x\to 3^+} \sqrt{9-x^2} = \lim_{x\to 3^+} \sqrt{-\underset{+}{(x-3)}(3+x)}$$

$x>3$
$x-3>0$

DNE

inv. $\sqrt{-(0^+)(6)} \to \sqrt{0^-}$ DNE

4

$$\therefore \boxed{\lim_{x\to 3} \sqrt{9-x^2} \quad \text{DNE}}$$

(d)

$$\lim_{x\to -3} \frac{x+3}{|x^2-9|} = \lim_{x\to -3} \frac{x+3}{|x+3|\,|x-3|}$$

inv. $\frac{0}{0}$ indeterminate, CZF $= x+3$

side limits:

$$\lim_{x\to -3^+} \frac{x+3}{|x^2-9|} = \lim_{x\to -3^+} \frac{x+3}{|x+3|\,|x-3|}$$

$x>-3$
$x+3>0$
$\Rightarrow |x+3| = x+3$

$$= \lim_{x\to -3^+} \frac{\cancel{x+3}}{\cancel{(x+3)}\,|x-3|}$$

$$= \frac{1}{6}$$

5

$$\lim_{x\to -3^-} \frac{x+3}{|x^2-9|} = \lim_{x\to -3^-} \frac{x+3}{|x+3|\,|x-3|}$$

$x<-3$
$x+3<0$
$\Rightarrow |x+3| = -(x+3)$

$$= \lim_{x\to -3^-} \frac{\cancel{x+3}}{-\cancel{(x+3)}\,|x-3|}$$

$$= -\frac{1}{6}$$

$$\lim_{x\to -3^+} \frac{x+3}{|x^2-9|} \neq \lim_{x\to -3^-} \frac{x+3}{|x^2-9|}$$

$$\therefore \boxed{\lim_{x\to -3} \frac{x+3}{|x^2-9|} \quad \text{DNE}}$$

(8) - continued

(e) $\lim_{x\to 0} \frac{\sin^3(3x)}{x^2 \sin(2x)} = \lim_{x\to 0} \frac{\sin 3x}{3x} \frac{\sin 3x}{3x} \frac{\sin 3x}{3x} \frac{2x}{\sin 2x} \frac{(3x)(3x)(3x)}{\cancel{x^2}(2x)}$

mv $\frac{0}{0}$ indeterminate

$= (1)(1)(1)(1)\ \frac{27}{2}$

(4)

$$\boxed{\lim_{x\to 0} \frac{\sin^3(3x)}{x^2 \sin(2x)} = \frac{27}{2}}$$

(f) $\boxed{\lim_{x\to -2} \frac{x^2-9}{\sqrt{(x+2)^2}} = -\infty}$

(4)

mv. $\frac{-5}{\sqrt{0^+}} \to \frac{-5}{0^+} \to -\infty$

(9) $f(x) = \frac{1}{x^2}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \cdot \frac{x^2(x+h)^2}{x^2(x+h)^2}$

mv. $\frac{0}{0}$ indeterminate CZF = h.

$= \lim_{h\to 0} \frac{x^2-(x+h)^2}{h\,x^2(x+h)^2}$

(6)

$= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{h\,x^2(x+h)^2}$

$= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}\,x^2(x+h)^2}$

$= \frac{-2\cancel{x}}{x^{\cancel{2}}(x^2)}$

$$\boxed{f'(x) = \frac{-2}{x^3}}$$

(10) (a) $f(x) = \frac{\sqrt[3]{x^2}}{x^2+1}$] D_x Quotient Rule.

D_x

$f'(x) = \frac{(x^{\frac{2}{3}})'(x^2+1) - (x^{\frac{2}{3}})(x^2+1)'}{(x^2+1)^2}$

(5)

$= \frac{\frac{2}{3}x^{-\frac{1}{3}}(x^2+1) - x^{\frac{2}{3}}(2x)}{(x^2+1)^2} \cdot \frac{3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$

$= \frac{2(x^2+1) - 6x^2}{3x^{\frac{1}{3}}(x^2+1)^2}$

$$\boxed{f'(x) = \frac{2-4x^2}{3\sqrt[3]{x}\,(x^2+1)^2}}$$

(10) – continued

(b) $y = \dfrac{2\cos x + \sin x - 3}{\sin x}$

$= \dfrac{2\cos x}{\sin x} + \dfrac{\sin x}{\sin x} - \dfrac{3}{\sin x}$

D_x ($y = 2\cot x + 1 - 3\csc x$) D_x

$\dfrac{dy}{dx} = -2\csc^2 x + 0 + 3\csc x \cot x$

$\boxed{\dfrac{dy}{dx} = -2\csc^2 x + 3\csc x \cot x}$

4

(11) $f(x) = \sqrt[3]{x}$

(a) $f'(x) = ?$

D_x ($f(x) = x^{\frac{1}{3}}$) D_x

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$\boxed{f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$}

3

(b) V.T.L. ?

Proof: $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0} \dfrac{1}{3\sqrt[3]{x^2}}$

inv. $\dfrac{1}{3\sqrt[3]{0^2}} \to \dfrac{1}{0^+} \to +\infty$

$\left.\begin{array}{l}\lim\limits_{x\to 0} f'(x) = \infty \\ x = 0 \in D_f\end{array}\right\} \Rightarrow \exists$ V.T.L. at $x = 0$

3

$\therefore$ $\boxed{\text{V.T.L. at } P(0,0)}$ Q.E.D.

(12) (Extra Credit)

$\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\sin x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}} = \lim\limits_{\theta\to 0} \dfrac{\sin(\theta + \frac{\pi}{4}) - \frac{\sqrt{2}}{2}}{\theta}$

inv. $\frac{0}{0}$ indeterminate. ZF=(x–

$\left\{\begin{array}{l}\text{let } \theta = x - \frac{\pi}{4} \Rightarrow x = \theta + \frac{\pi}{4} \\ \text{as } x \to \frac{\pi}{4} \Rightarrow \theta \to 0\end{array}\right.$

$= \lim\limits_{\theta\to 0} \dfrac{\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} - \frac{\sqrt{2}}{2}}{\theta}$

$= \left(\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}\right)\cdot\dfrac{\sqrt{2}}{2} + \lim\limits_{\theta\to 0} \dfrac{\cos\theta\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}}{\theta}$

+5

$= \dfrac{\sqrt{2}}{2}\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} + \dfrac{\sqrt{2}}{2}\lim\limits_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta}$

$= \dfrac{\sqrt{2}}{2}(1) + \dfrac{\sqrt{2}}{2}(0)$

$\boxed{\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\sin x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}} = \dfrac{\sqrt{2}}{2}}$