Math 7 **Exam #2** **Name:** Key

Use **both sides** of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline that was requested for the homework assignments. Also note that you can **only use formulas and methods that were presented** thus far **in the lecture**.

1. Given $x^2 - y^2 = 9$. Use implicit differentiation to find each of the followings:

a. $\dfrac{dy}{dx}$ **b.** $\dfrac{d^2y}{dx^2}$ *(simplify completely)*

2. Evaluate each of the following limits:

a. $\displaystyle\lim_{x\to-\infty} \frac{3x-5}{2x-\sqrt{3x^2+x-1}}$ **b.** $\displaystyle\lim_{\theta\to\infty} \frac{2\theta+\sin\theta}{\cos\theta-3\theta}$

3. State each of the following definitions or theorems:

a. definition of relative/local maximum point

b. definition of a decreasing function on an interval I

c. Mean Value Theorem of derivative (*also show a graphical illustration*)

4. Use the second derivative test to sketch the graph of $f(x) = \sqrt[3]{x}(x-1)$. (*number your steps and show all details as illustrated during the lecture*) $\frac{1}{2}$

5. Use the 8-legendary steps to sketch the graph of the following function with the give 1st and 2nd derivatives. (*Number your steps and show all details as illustrated in the lecture*) $\frac{(\sqrt{3}+1)^2}{4}$

$$f(x) = \frac{(x+1)^3}{(1+x^2)(x+1)},\quad f'(x) = \frac{2(1-x^2)}{(1+x^2)^2},\quad f''(x) = \frac{4x(x^2-3)}{(1+x^2)^3}$$

6. A right circular cone whose altitude is twice its base radius is circumscribed over a right circular cylinder with a constant height of 5 inches. The altitude of the cone is decreasing at the rate of 2 inches per minute. At what rate is the radius of the cylinder changing when the altitude of the cone is 10 inches? $\left(\frac{\sqrt{3}}{2}\right)^2$ $\frac{3}{4}$

7. An airplane is flying horizontally at a constant speed passes through a point 10,000 feet directly above an observer on the ground. A moment later, the observer notices that the angle of elevation of the plane is changing at the rate of 1 degree per minute when the angle of elevation of the plane is $60°$. Find the speed of the plane.

Evaluate each of the following integrals:

8. $\displaystyle\int \frac{2\sin^2 x - 3\cos^2 x}{\cos^2 x}\,dx$ **9.** $\displaystyle\int \frac{\sec\left(\sqrt[3]{x}+1\right)\tan\left(\sqrt[3]{x}+1\right)}{\sqrt[3]{x^2}}\,dx$

10. $\displaystyle\int \left(\sqrt[3]{x}-1\right)^{100} dx$ **11.** $\displaystyle\int \tan^3\theta \sec^5\theta \, d\theta$

12. (Extra Credit) A sphere is inscribed in a cone whose base radius is 5 inches. The sphere's radius is increasing at the rate of 2 inches per min. At what rate is the altitude of the cone changing when the radius of the sphere is 3 inches?

① $x^2 - y^2 = 9$ ⓐ

$$D_x\left(x^2 - y^2 = 9\right)D_x$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\boxed{\frac{dy}{dx} = \frac{x}{y}}$$ (4)

ⓑ $D_x\left(\frac{dy}{dx} = \frac{x}{y}\right)D_x$

$$\frac{d^2y}{dx^2} = \frac{1y - x\frac{dy}{dx}}{y^2}$$

$$= \frac{y - x\frac{x}{y}}{y^2}\cdot\frac{y}{y}$$ (6)

$$= \frac{y^2 - x^2}{y^3}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-9}{y^3}}$$

② ⓐ $\lim\limits_{x\to-\infty} \dfrac{3x-5}{2x-\sqrt{3x^2+x-1}} = \lim\limits_{x\to-\infty} \dfrac{x\left[3-\frac{5}{x}\right]}{2x-|x|\sqrt{3+\frac{1}{x}-\frac{1}{x^2}}}$

inv. $\dfrac{\infty}{-\infty-\sqrt{\infty-\infty}}$ ← indeterminate

$x<0 \Rightarrow |x| = -x$

$$= \lim_{x\to-\infty} \frac{x\left[3-\frac{5}{x}\right]}{2x+x\sqrt{3+\frac{1}{x}-\frac{1}{x^2}}}$$ (6)

$$= \lim_{x\to-\infty} \frac{\cancel{x}\left(3-\frac{5}{x}\right)}{\cancel{x}\left[2+\sqrt{3+\frac{1}{x}-\frac{1}{x^2}}\right]}$$

inv. $\dfrac{3-\frac{5}{-\infty}\to 0}{2+\sqrt{3+\frac{1}{-\infty}+\frac{1}{\infty}}}$

$$\boxed{\lim_{x\to-\infty} \frac{3x-5}{2x-\sqrt{3x^2+x-1}} = \frac{3}{2+\sqrt{3}}}$$

ⓑ $\lim\limits_{\theta\to\infty} \dfrac{2\theta+\sin\theta}{\cos\theta-3\theta} = \lim\limits_{\theta\to\infty} \dfrac{\cancel{\theta}\left[2+\frac{\sin\theta}{\theta}\right]}{\cancel{\theta}\left[\frac{\cos\theta}{\theta}-3\right]}$

inv $\dfrac{\infty}{\infty}$ indeterminate

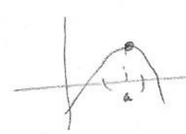
inv $\dfrac{2+\frac{\sin\theta}{\infty}\to 0}{\frac{\cos\theta}{\infty}\to 0 - 3}$ sin θ & cos θ are finite

$$\boxed{\lim_{\theta\to\infty} \frac{2\theta+\sin\theta}{\cos\theta-3\theta} = -\frac{2}{3}}$$ (5)

③ ⓐ Def.: local max pt. If ∃ an open interval I containing "a" ST.

$$f(a) \geq f(x), \quad \forall x \in I$$

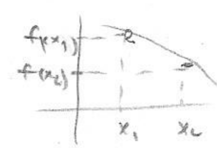
Then $P(a, f(a))$ is s.t.b. a local / relative max pt.



(4)

ⓑ Def.: dec. func. on I: If $f(x_1) > f(x_2)$, $\forall\, x_1, x_2 \in I$ where $x_1 < x_2$

Then $f(x)$ is s.t.b. a dec. func. on I.



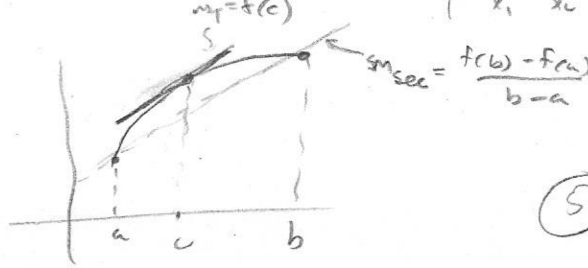
(4)

ⓒ M.V.T.: If ① $f(x)$ is cont. on $[a, b]$

& ② $f(x)$ is diff. on (a, b)

Then ∃ $c \in (a, b)$ S.T.

$$\underbrace{f'(c)}_{m_T} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{m_{sec}}$$



(5)

④ 2nd Deriv Test $f(x) = \sqrt[3]{x}\,(x-1)$

① $D_f = \mathbb{R}$

② $\begin{cases} x\text{-int}: & x=0,\ x=1 \\ y\text{-int}: & y=0 \end{cases}$

③ $f'(x) = \left[\frac{1}{3}x^{-\frac{2}{3}}(x-1) + x^{\frac{1}{3}}\right] \cdot \frac{3x^{\frac{2}{3}}}{3x^{\frac{2}{3}}}$

$= \frac{(x-1)+3x}{3x^{\frac{2}{3}}}$

$\boxed{f'(x) = \frac{4x-1}{3x^{\frac{2}{3}}}}$ $\xrightarrow[\text{H.T.L}]{=0}$ $x = \frac{1}{4}$; $\xrightarrow[\text{V.T.L}]{\text{DNE}}$ $x=0$

④ $f''(x) = \frac{1}{3}\,\frac{4x^{\frac{2}{3}} - (4x-1)\frac{2}{3}x^{-\frac{1}{3}}}{x^{\frac{4}{3}}} \cdot \frac{3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$

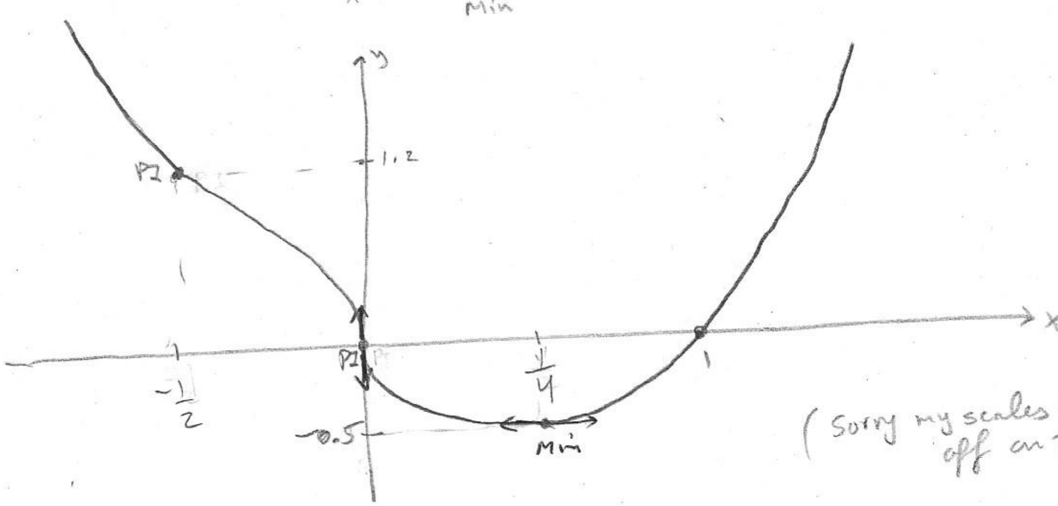
$= \frac{1}{9}\,\frac{12x - 2(4x-1)}{x^{\frac{5}{3}}}$ $-4x+2$

$\boxed{f''(x) = \frac{1}{9}\,\frac{4x+2}{x^{\frac{5}{3}}}}$ $\xrightarrow{=0}$ $x = -\frac{1}{2}$; $\xrightarrow{\text{DNE}}$ $x=0$

⑤

x	$-\infty$		$-\frac{1}{2}$		0		$\frac{1}{4}$		$+\infty$
$f'(x)$		$-$		$-$	//	$-$	0	$+$	
$f''(x)$		$+$	0	$-$	//	$+$		$+$	
$f(x)$		↘	1.2 PI	↘	0 V.T.L PI	↘	-0.5 H.T.L Min	↗	

⑥



(Sorry my scales are off on the x-axis!)

⑩

⑤ $f(x) = \dfrac{(x+1)^3}{(1+x^2)(x+1)}$, $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2}$, $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3}$

① $\boxed{f(x) = \dfrac{(x+1)^2}{(1+x^2)}}$, $x \neq -1$ hole

$D_f = \mathbb{R} - \{-1\}$

② $\begin{cases} x\text{-int: } x = -1 \leftarrow \text{note hole} \\ y\text{-int: } y = 1 \end{cases}$

③ VA: None

(12)

④ HA: $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{(x+1)^2}{1+x^2}$

$= \lim\limits_{x \to \pm\infty} \dfrac{\cancel{x^2}(1+\frac{1}{x})}{\cancel{x^2}(\frac{1}{x^2}+1)}$

mm $\dfrac{1+\cancel{\frac{1}{\infty}}}{\cancel{\frac{1}{\infty}}+1} \to 1$

$\boxed{\lim\limits_{x \to \pm\infty} f(x) = 1} \Rightarrow \exists$ HA at $y = 1$ when $x \to +\infty$ & $x \to -\infty$

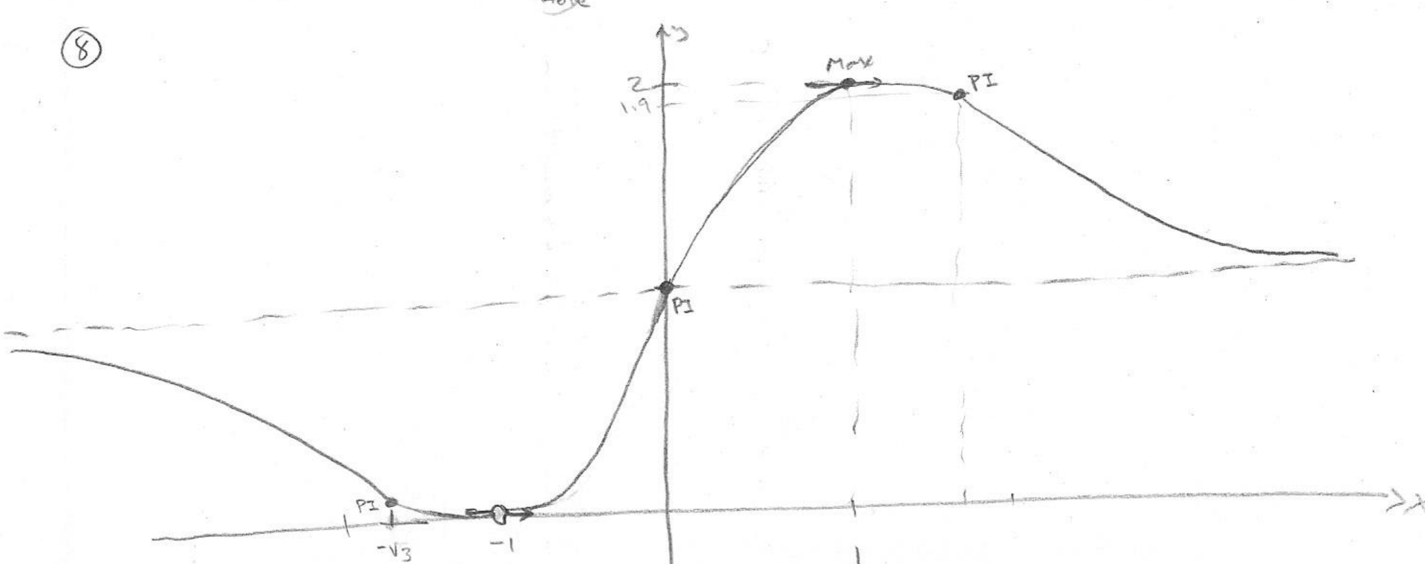
⑤ $f'(x) = \dfrac{2(1-x^2)}{(1+x^2)^2}$ — $\xrightarrow[\text{H.T.L}]{=0} x = \pm 1$; $\xrightarrow[\text{V.T.L. VA}]{\text{DNE}}$ none

⑥ $f''(x) = \dfrac{4x(x^2-3)}{(1+x^2)^3}$ — $\xrightarrow{=0} x = 0,\ x = \pm\sqrt{3}$; $\xrightarrow{\text{DNE}}$ none

⑦

x	$-\infty$		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		∞
$f'(x)$		$-$		$-$	0	$+$		$+$	0	$-$		$-$	
$f''(x)$		$-$	0	$+$		$+$	0	$-$		$-$	0	$+$	
$f(x)$	1 HA	↘	0.1 ~~P~~PI	↘	0 HTL Min Hole	↗	1 ~~P~~PI	↗	2 H.T.L Max	↘	1.9 ~~P~~PI	↘	1 HA

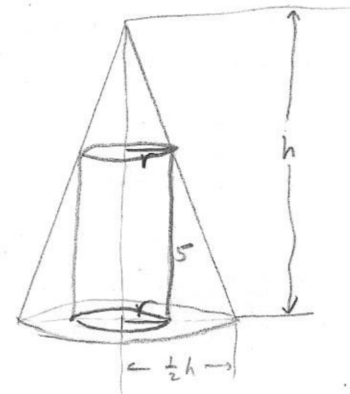
⑧



(6)

$$\frac{dh}{dt} = -2 \ \frac{\text{inch}}{\text{min}}$$

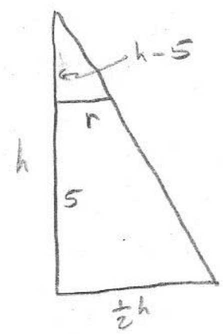
$$\left.\frac{dr}{dt}\right|_{h=10''} = ?$$



(8)

Similar Δ's:

$$\frac{\frac{1}{2}h}{r} = \frac{h}{h-5}$$



$$r = \frac{1}{2}(h-5) \quad \Big) D_t$$

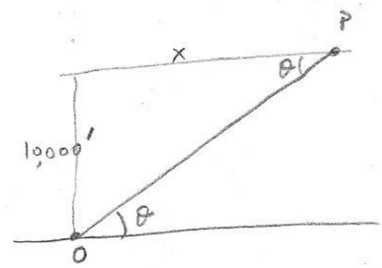
$$D_t \Big(\quad \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt} \rightarrow -2$$

$$\boxed{\left.\frac{dr}{dt}\right|_{h=10''} = -1 \ \frac{\text{inch}}{\text{min}}}$$

(7)

$$\left.\frac{d\theta}{dt}\right|_{\theta=60^\circ} = -1 \ \frac{\text{deg}}{\text{min}}\left(\frac{\pi}{180}\ \frac{\text{Rad.}}{\text{deg.}}\right)$$

$$= \frac{-\pi}{180}\ \frac{\text{Rad}}{\text{min}}$$



$$\left.\frac{dx}{dt}\right|_{\theta=60^\circ} = ?$$

$$\text{Tan}\,\theta = \frac{10000}{x}$$

$$x = \frac{10000}{\text{Tan}\,\theta}$$

(8)

$$x = 10000 \cot\theta \quad \Big) D_t$$

$$D_t \Big(\quad \frac{dx}{dt} = 10000(-\csc^2\theta)\frac{d\theta}{dt} \rightarrow \frac{-\pi}{180}$$

$$\left.\frac{dx}{dt}\right|_{\theta=60^\circ} = 1000\cancel{0}(-\csc^2 60^\circ)\left(\frac{-\pi}{18\cancel{0}}\right)$$

$$= 1000\left(-\frac{4}{3}\right)\left(\frac{-\pi}{18}\right)$$

$$\boxed{\frac{dx}{dt} = \frac{2000\pi}{27}\ \frac{ft}{min}}$$

(8) $I = \int \frac{2\sin^2 x - 3\cos^2 x}{\cos^2 x}\,dx$

$= \int \frac{2(1-\cos^2 x) - 3\cos^2 x}{\cos^2 x}\,dx$

(7)

$= \int \frac{2 - 5\cos^2 x}{\cos^2 x}\,dx$

$= \int (2\sec^2 x - 5)\,dx$

$\boxed{I = 2\tan x - 5x + C}$

(9) $I = \int \frac{\sec(\sqrt[3]{x}+1)\tan(\sqrt[3]{x}+1)}{\sqrt[3]{x^2}}\,dx$

Goal: $\int \sec u \tan u\,du = \sec u + C$

let $u = \sqrt[3]{x} + 1$

$3\,du = \frac{1}{3} x^{-\frac{2}{3}}\,dx$

$= \int \sec(\sqrt[3]{x}+1)\tan(\sqrt[3]{x}+1)\; x^{-\frac{2}{3}}\,dx$

(7)

$= \int \sec u \tan u \; 3\,du$

$= 3\sec u + C$

$\boxed{I = 3\sec(\sqrt[3]{x}+1) + C}$

Goal: $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$

let $u = \sqrt[3]{x} - 1 \Rightarrow x = (u+1)^3$

$3\,du = \frac{1}{3}x^{-\frac{2}{3}}\,dx$

(10) $I = \int (\sqrt[3]{x} - 1)^{100}\,dx$

$= \int x^{\frac{2}{3}}(x^{\frac{1}{3}} - 1)^{100}\; x^{-\frac{2}{3}}\,dx$

(7)

$= \int \left[(u+1)^3\right]^{\frac{2}{3}} u^{100}\; 3\,du$

$= 3\int (u^2 + 2u + 1)u^{100}\,du$

$= 3\int (u^{102} + 2u^{101} + u^{100})\,du$

$= 3\left[\frac{1}{103}u^{103} + \frac{2}{102}u^{102} + \frac{1}{101}u^{101}\right] + C$

$\boxed{I = \frac{3}{103}(\sqrt[3]{x}-1)^{103} + \frac{1}{17}(\sqrt[3]{x}-1)^{102} + \frac{3}{101}(\sqrt[3]{x}-1)^{101} + C}$

Goal: $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$

let $u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

(11) $I = \int \tan^3\theta \sec^5\theta\,d\theta$

$= \int \underbrace{\tan^2\theta}_{(\sec^2\theta - 1)} \sec^4\theta \; \sec\theta\tan\theta\,d\theta$

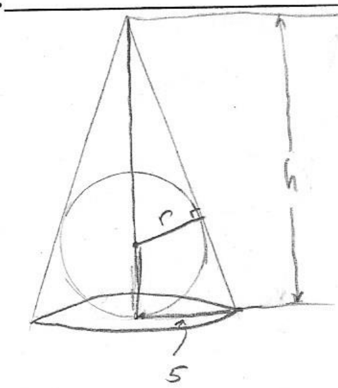
(7)

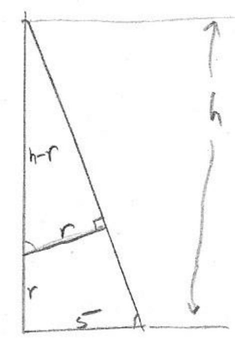
$= \int (\sec^6\theta - \sec^4\theta)\sec\theta\tan\theta\,d\theta$

$= \int (u^6 - u^4)\,du$

$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$

$\boxed{I = \frac{1}{7}\sec^7\theta - \frac{1}{5}\sec^5\theta + C}$





(12) (Extra Credit)

$$\frac{dr}{dt} = +2 \ \frac{\text{in}}{\text{min}}$$

$$\left.\frac{dh}{dt}\right|_{r=3} = ?$$

similar Δ's:

$$\cancel{\frac{h}{?}} = \frac{5}{r} = \frac{\sqrt{h^2+25}}{h-r}$$

$$5(h-r) = r\sqrt{h^2+25}$$

$$25(h-r)^2 = r^2(h^2+25)$$

$$\frac{25h^2}{h} - \frac{50hr}{h} + \cancel{25r^2} = \frac{r^2h^2}{h} + \cancel{25r^2}$$

$$25h - 50r = r^2 h$$

$$25h - r^2 h = 50r$$

$$(25-r^2)h = 50r$$

$$D_t\left(\ \boxed{h = \frac{50r}{25-r^2}}\ \right) D_t$$

$$\frac{dh}{dt} = 50\,\frac{\frac{dr}{dt}(25-r^2) - r(-2r)\frac{dr}{dt}}{(25-r^2)^2}$$

$$= 50\,\frac{25-r^2+2r^2}{(25-r^2)^2}\,\frac{dr}{dt} \rightarrow 2$$

$$\boxed{\frac{dh}{dt} = 100\,\frac{25+r^2}{(25-r^2)^2}}$$

$$\left.\frac{dh}{dt}\right|_{r=3} = 100\,\frac{25+9}{(25-9)^2}$$

$$= \frac{3400}{256}$$

$$\boxed{\left.\frac{dh}{dt}\right|_{r=3} = \frac{425}{32}\ \frac{\text{in}}{\text{min}}}$$

$\boxed{+5}$