Math 7 **Exam #3** **Name:** Key

Use **both sides** of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline that was requested for the homework assignments. Also note that you can **only use formulas and methods that were presented** thus far **in the lecture**.

Evaluate each of the followings:

1. $\int_0^2 x^3 \sqrt[3]{x^2+4}\,dx$
2. $\int_{-3}^0 -\sqrt{9-x^2}\,dx$
3. $\int_{-1}^3 |x-2|\,dx$
4. Prove the Fundamental Theorem of Calculus Part I: f is a continuous function on a closed interval $[a,b]$. If the function G is defined as $G(x)=\int_a^x f(t)\,dt,\ \forall x\in[a,b]$, then G is an antiderivative of f.
5. Prove the Fundamental Theorem of Calculus Part II: If F is an antiderivative of f, then $\int_a^b f(t)\,dt = F(b)-F(a)$.
6. Find the dimensions of the largest right circular cone inscribed in a sphere with a constant radius of 10 inches.
7. A vertical fence 8 feet tall stands on a level ground 1 foot away and parallel to a vertical tall building. A ladder extends from the ground over the fence to the wall of the building. Find the shortest such ladder.
8. Use the right-end method to find the area of $R_{ty}=\left\{y=t^3, t-axis, t\in[0,x]\right\}$. $x>0$
9. Given $R_{xy}=\left\{y=f(x), y=g(x), x\in[a,b]\right\}$ for $f(x)\le g(x)$. Prove that the area of the R_{xy} is $A=\int_a^b\left[g(x)-f(x)\right]dx$.
10. Given $R_{xy}=\left\{y=x^2, y=\sqrt{x}, x\in[0,2]\right\}$. Find the area of the enclosed region.
11. Given $R_{xy}=\left\{y=x^2, y=\sqrt{x}, x\in[0,2]\right\}$. Setup (but don't evaluate) the integral(s) with respect to y representing the area of R_{xy}.
12. Given $R_{xy}=\left\{y=\sqrt{x}+1, y=2x, y=1\right\}$ revolved about $y=1$. Setup (don't evaluate) the integral(s) representing the volume of the solid using disk and/or washer.
13. Given $R_{xy}=\left\{y=x^2, y=x+2\right\}$ revolved about $x=-2$. Setup (don't evaluate) the integral(s) representing the volume of the solid using disk and/or washer.
14. Use the methods of solids due to revolution to prove the volume formula for a cone with altitude of H and base radius of R.
15. **(Extra Credit)** Evaluate the following integral. (Even though 0 is not in the domain of the following function, later in math 8 we learn how to deal with these types of integral. For now, ignore the fact the 0 in not in the domain and use some kind of substitution to integrate. After the substitution, this undefined situation will magically disappear!)

$$\int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}}\,dx$$

(1) $I = \int_0^2 x^3 \sqrt[3]{x^2+4}\, dx$

Goal: $\int u^n\, du = \frac{1}{n+1} u^{n+1} + C$

let $u = x^2+4 \Rightarrow x^2 = u-4$

$\frac{1}{2} du = \cancel{2}x\, dx$

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$= \int_0^2 x^3 (x^2+1)^{\frac{1}{3}}\, dx$

$= \int_0^2 x^2 (x^2+1)^{\frac{1}{3}}\; x\,dx$

$= \int_a^b (u-4)\, u^{\frac{1}{3}}\, \frac{1}{2} du$

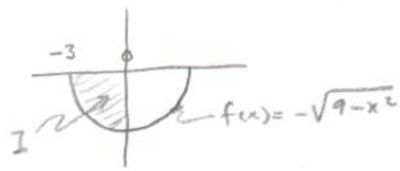
$= \frac{1}{2} \int_a^b (u^{\frac{4}{3}} - 4u^{\frac{1}{3}})\, du$

$= \frac{1}{2} \left[\frac{3}{7} u^{\frac{7}{3}} - \cancel{4}(\frac{3}{\cancel{4}}) u^{\frac{4}{3}} \right]_a^b$

$= \frac{1}{2} \frac{1}{7} \left[3(x^2+4)^{\frac{7}{3}} - 21 (x^2+4)^{\frac{4}{3}} \right]_0^2$

$$\boxed{I = \frac{1}{14}\left[\left(3(2)^7 - 21(2)^4\right) - \left(3(4)^{\frac{7}{3}} - 21\,(4)^{\frac{4}{3}}\right)\right]}$$

(2) $I = \int_{-3}^{0} -\sqrt{9-x^2}\, dx$



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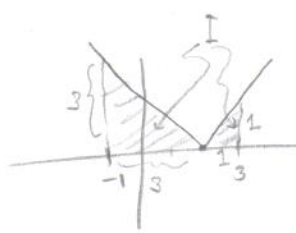
$= \frac{-1}{4} \pi (3)^2$

$\boxed{I = \frac{-9\pi}{4}}$

(3) $I = \int_{-1}^{3} |x-2|\, dx$



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$= \frac{1}{2}(3)(3) + \frac{1}{2}(1)(1)$

$= \frac{9}{2} + \frac{1}{2}$

$\boxed{I = 5}$

(4) F.T.C Pf:

Proof: Given $G(x) = \int_a^x f(t)\,dt$, $\forall x \in [a,b]$ find $G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h}$

$$G(x+h) = \int_a^{x+h} f(t)\,dt$$

$$-\quad G(x) = \int_a^{x} f(t)\,dt$$

$$G(x+h) - G(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$$

$$= \int_a^{x+h} f(t)\,dt + \int_x^{a} f(t)\,dt$$

$$= \int_x^{a} f(t)\,dt + \int_a^{x+h} f(t)\,dt$$

$$\boxed{G(x+h) - G(x) = \int_x^{x+h} f(t)\,dt} \quad (1)$$

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Given $f(x)$ is continuous on $[x, x+h]$ (for $h>0$) (or $[x+h, x]$ for $h<0$)

$\Rightarrow$ by M.V.T. $\exists\, z \in (x, x+h)$ s.t. (or $z \in (x+h, x)$)

$$\int_x^{x+h} f(t)\,dt = f(z)\,(x+h-x)$$

$$\boxed{\int_x^{x+h} f(t)\,dt = h\,f(z)\ ;\quad x < z < x+h \quad (\text{or } x+h < z < x)} \quad (2)$$

Subst. (2) into (1):

$$\left.\begin{aligned} &\Rightarrow G(x+h) - G(x) = h\,f(z)\ ;\ x<z<x+h \\ &G'(x) = \lim_{h\to 0} \frac{G(x+h)-G(x)}{h} \\ &\text{as } h \to 0 \Rightarrow z \to x \end{aligned}\right\} \Rightarrow G'(x) = \lim_{z\to x} \frac{\cancel{h} f(z)}{\cancel{h}}$$

$$\Rightarrow \quad G'(x) = f(x)$$

$\therefore$ $\boxed{G \text{ is an antideriv. of } f}$ Q.E.D

(5) F.T.C P II:

Proof: Given F be an antideriv. of f.

let $G(x)=\int_a^x f(t)\,dt \Rightarrow$ by F.T.C P I, G is an antideriv. of f

$\Rightarrow$ F & G are both antideriv. of f.

$\Rightarrow$ F & G differ by a constant

$\Rightarrow \quad G(x) = F(x) + C$

$$\boxed{\int_a^x f(t)\,dt = F(x) + C \quad , \forall x \in [a,b]} \quad (1)$$

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let $x=a$:

$$\underbrace{\int_a^a f(t)\,dt}_{0} = F(a) + C$$

$$0 = F(a) + C \Rightarrow \boxed{C = -F(a)}$$

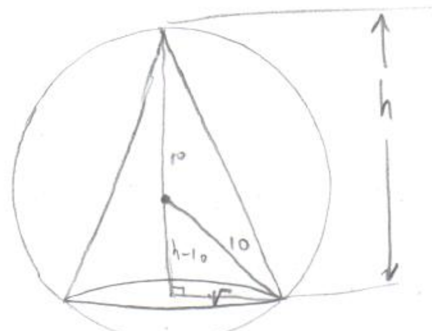
$$\Rightarrow \int_a^x f(t)\,dt = F(x) - F(a) \quad \forall x \in [a,b]$$

let $x=b$: $\quad \therefore \boxed{\int_a^b f(t)\,dt = F(a) - F(b)}$ Q.E.D.

(6) Objective: let V= volume of cone

$V \to$ Max.

$\boxed{V = \frac{1}{3}\pi r^2 h}$ Pending



Constraint: sphere w/ radius 10".

$$(h-10)^2 + r^2 = 10^2$$

$$h^2 - 20h + \cancel{100} + r^2 = \cancel{100}$$

$$\boxed{r^2 = 20h - h^2}$$

$$V(h) = \frac{1}{3}\pi(20h - h^2)h$$

$$= \frac{1}{3}\pi(20h^2 - h^3) \quad \} D_h$$

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D_h

$$V'(h) = \frac{1}{3}\pi(40h - 3h^2) \quad \xrightarrow{=0} h=0,\ h=\frac{40}{3}$$

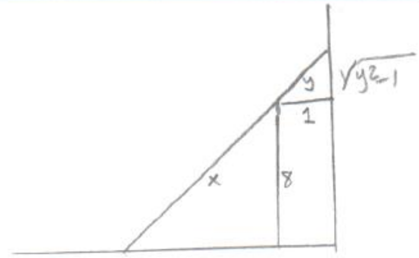
$$= \frac{1}{3}\pi h(40 - 3h) \quad \text{DNE} \Rightarrow \text{None}$$

h	0		$\frac{40}{3}$		20
$V'(h)$		+	0	—	
$V(h)$		↗	Max	→	

$V \to$ max when $h = \frac{40}{3}$, $r = ?$

$$r = \sqrt{20\left(\frac{40}{3}\right) - \left(\frac{40}{3}\right)^2} \Rightarrow r = \sqrt{\frac{800}{9}} \Rightarrow \boxed{r = \frac{20\sqrt{2}}{3}}$$

$\Rightarrow$ $\boxed{V \to \text{Max when } h = \frac{40}{3}\text{" } \& \ r = \frac{20\sqrt{2}}{3}\text{"}}$



(7) <u>objective</u>: let l = length of ladder

$l \to \text{Min}$

$\boxed{l = x+y}$ Pending

<u>Constraint</u>: 8-ft fence, 1ft away from Building.

similar Δ's: $\frac{x}{y} = \frac{8}{\sqrt{y^2-1}} = \cancel{?}$

$x\sqrt{y^2-1} = 8y$

$\boxed{x = \frac{8y}{\sqrt{y^2-1}}}$

8

$D_y\left(\; l(y) = \frac{8y}{\sqrt{y^2-1}} + y \; \right) D_y$

$l'(y) = 8\,\frac{1(y^2-1)^{\frac{1}{2}} - y\frac{1}{2}(y^2-1)^{-\frac{1}{2}}(2y)}{y^2-1} \cdot \frac{(y^2-1)^{\frac{1}{2}}}{(y^2-1)^{\frac{1}{2}}} + 1$

$= 8\,\frac{\cancel{y^2}-1-\cancel{y^2}}{(y^2-1)^{\frac{3}{2}}} + 1$

$\boxed{l'(y) = \frac{-8 + (y^2-1)^{\frac{3}{2}}}{(y^2-1)^{\frac{3}{2}}}}$

$\xrightarrow{=0} (y^2-1)^{\frac{3}{2}} = 8 \Rightarrow y^2-1 = 4 \Rightarrow y^2 = 5 \Rightarrow y = \pm\sqrt{5}$

$\xrightarrow{DNE} y = 1, \cancel{-1}$

y	1		$\sqrt{5}$		$+\infty$
$l'(y)$		—	0	+	
$l(y)$		↘	Min.	↗	

$\Rightarrow l \to \text{min}$ when $\boxed{y = \sqrt{5}} \Rightarrow x = \frac{8\sqrt{5}}{\sqrt{4}}$

$\boxed{x = 4\sqrt{5}}$

$\Rightarrow l = \sqrt{5} + 4\sqrt{5}$

$= 5\sqrt{5}$

$\therefore \boxed{l \to \text{Min when } l = 5\sqrt{5} \text{ ft}}$

(8) $R_{xy} = \{ y = t^3, T\text{-axis}, T \in [0, x] \}$

Estimation;

$\boxed{w_i = \Delta t, \quad \Delta t = \frac{x}{n}}$

$l_i = f(a + i\Delta t) \Rightarrow l_i = f(i(\Delta t))$

$= (i(\Delta t))^3$

$\boxed{l_i = (\Delta t)^3 i^3}$

$A_i = l_i w_i$

$= (\Delta t)^3 i^3 (\Delta t)$

$= (\Delta t)^4 i^3$

$A \approx \sum_{i=1}^{n} A_i$

$\approx \sum_{i=1}^{n} (\Delta t)^4 i^3$

$\approx (\Delta t)^4 \sum_{i=1}^{n} i^3$

$\approx (\Delta t)^4 \left(\frac{n(n+1)}{2}\right)^2$

$\approx \left(\frac{x}{n}\right)^4 \frac{n^2(n+1)^2}{4}$

$\boxed{A \approx \frac{x^4}{n^{\cancel{4}2}} \frac{\cancel{n^2}(n+1)^2}{4}}$

Refinement: $A = \lim_{n\to\infty} x^4 \frac{(n+1)^2}{4n^2}$

$= \frac{x^4}{4} \lim_{n\to\infty} \frac{(n+1)^2}{n^2}$

$= \frac{x^4}{4} \left(\lim_{n\to\infty} \frac{n^2 + \cdots}{n^2}\right) \to 1$

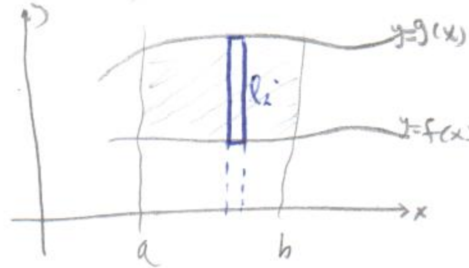
$\boxed{A = \frac{1}{4} x^4}$

(8)

Proof: Given

(9) $R_{xy} = \{ y = f(x),\ y = g(x),\ x \in [a,b] \},\quad f(x) \le g(x)$

Estimation: $x \in [a,b]$



$w_i = \Delta x_i$

vertical

$l_i = y_{Top} - y_{bottom}$

$l_i = g(x_i^*) - f(x_i^*)$

$A_i = \underbrace{[g(x_i^*) - f(x_i^*)]}_{l_i} \underbrace{\Delta x_i}_{w_i}$

$A \approx \sum_{i=1}^{n} A_i$

$\boxed{A \approx \sum_{i=1}^{n} [g(x_i^*) - f(x_i^*)] \Delta x_i, \quad x \in [a,b]}$

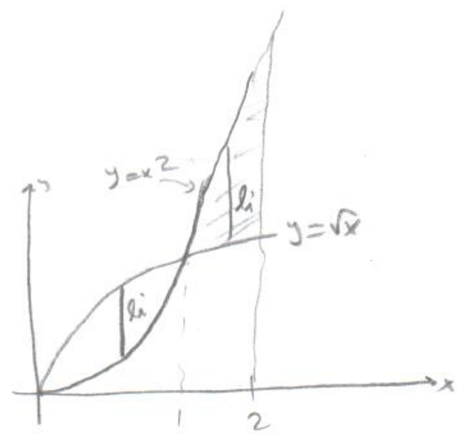
(6)

Refinement: $A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} [g(x_i^*) - f(x_i^*)] \Delta x_i, \quad x \in [a,b]$

$\max \Delta x_i \to 0$

$\therefore \boxed{A = \int_a^b [g(x) - f(x)]\,dx}$ Q.E.D

(10) $R_{xy} = \{ y = x^2,\ y = \sqrt{x},\ x \in [0,2] \}$ Find area



width $= dx$

Region ①: $x \in [0,1]$

vertical

$l_i \overset{\downarrow}{=} y_{Top} - y_{bottom}$

$= \sqrt{x} - x^2$

$A_1 = \int_0^1 (\sqrt{x} - x^2)\,dx$

$= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \Big]_0^1$

$= \frac{1}{3}[(2-1) - 0]$

$\boxed{A_1 = \frac{1}{3}}$

(6)

Region ②: $x \in [1,2]$

vertical

$l_i \overset{\downarrow}{=} y_{Top} - y_{bottom}$

$= x^2 - \sqrt{x}$

$A_2 = \int_1^2 (x^2 - \sqrt{x})\,dx$

$= \frac{1}{3}x^3 - \frac{2}{3}x^{\frac{3}{2}} \Big]_1^2$

$= \frac{1}{3}[(8 - 2(2)^{\frac{3}{2}}) - (1-2)]$

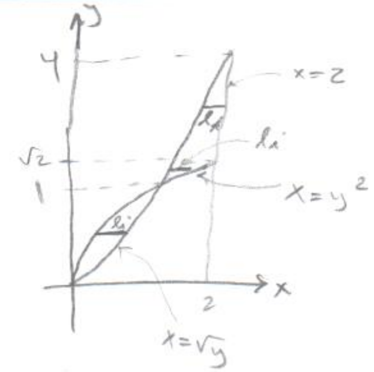
$= \frac{1}{3}(8 - 4\sqrt{2} + 1) \Rightarrow \boxed{A_2 = \frac{1}{3}(9 - 4\sqrt{2})}$

$A = A_1 + A_2$

$A = \frac{1}{3} + \frac{1}{3}(9 - 4\sqrt{2})$

$\boxed{A = \frac{10 - 4\sqrt{2}}{3}}$

(11) $R_{xy} = \{ y = x^2, y = \sqrt{x}, x \in [0, 2] \}$. A in Terms of y.



width $= dy$

Region ①: $y \in [0, 1]$

$l_i \overset{\text{horiz.}}{=} x_R - x_L$

$= \sqrt{y} - y^2$

$$\boxed{A_1 = \int_0^1 (\sqrt{y} - y^2)\, dy}$$

Region ②: $y \in [1, \sqrt{2}]$

$l_i \overset{\text{horiz.}}{=} x_R - x_L$

$l_i = y^2 - \sqrt{y}$

$$\boxed{A_2 = \int_1^{\sqrt{2}} (y^2 - \sqrt{y})\, dy}$$

Region 3: $y \in [\sqrt{2}, 4]$

$l_i \overset{\text{horiz.}}{=} x_R - x_L$

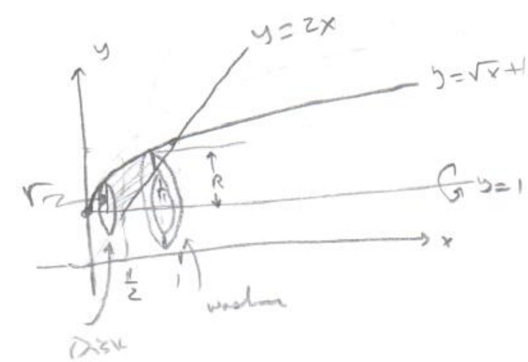
$= 2 - \sqrt{y}$

$$\boxed{A_3 = \int_{\sqrt{2}}^4 (2 - \sqrt{y})\, dy}$$

$\Rightarrow \boxed{A = A_1 + A_2 + A_3}$

(7)

(12) $R_{xy} = \{ y = \sqrt{x} + 1, y = 2x, y = 1 \}$ about $y = 1$



Thickness $= dx$

Region ①: $x \in [0, \frac{1}{2}]$

Disk: Area Disk $= \pi r^2$

$r \overset{\text{vertical}}{=} y_{Top} - y_{Bottom}$

$= (\sqrt{x} + 1) - 1$

$r = \sqrt{x}$

$$\boxed{V_1 = \int_0^{\frac{1}{2}} \pi (\sqrt{x})^2\, dx}$$

(6)

Region ②: $x \in [\frac{1}{2}, 1]$

washer: area $= \pi (R^2 - r^2)$

$R \overset{\text{vertical}}{=} y_{Top} - y_{Bottom}$

$= (\sqrt{x} + 1) - 1$

$\boxed{R = \sqrt{x}}$

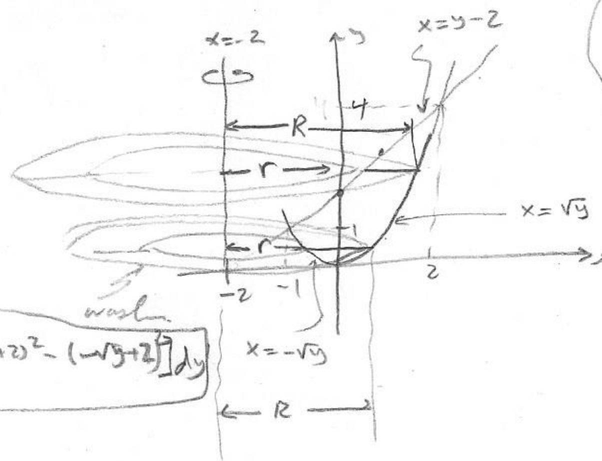
$r \overset{\text{vertical}}{=} y_{Top} - y_{Bottom}$

$\boxed{r = 2x - 1}$

$\Rightarrow \boxed{V_2 = \int_{\frac{1}{2}}^1 \pi \left[(\sqrt{x})^2 - (2x - 1)^2 \right] dx}$

$\boxed{V = V_1 + V_2}$

(13) $R_x = \{ y = x^2,\ y = x+2 \}$ about $x = -2$



$x^2 \stackrel{?}{=} x+2$
$x^2 - x - 2 = 0$
$(x-2)(x+1) = 0$
$x = 2,\ x = -1$

Thickness $= dy$

Region (1) : $y \in [0, 1]$

washer $\Rightarrow$ area $= \pi [R^2 - r^2]$

$R \stackrel{\text{Horiz.}}{=} x_R - x_L$

$= \sqrt{y} - (-2)$

$\boxed{R = \sqrt{y} + 2}$

$r \stackrel{\text{Horiz}}{=} x_R - x_L$

$= -\sqrt{y} - (-2)$

$\boxed{r = -\sqrt{y} + 2}$

$\Rightarrow \boxed{V_1 = \int_0^1 \pi \left[(\sqrt{y}+2)^2 - (-\sqrt{y}+2)^2 \right] dy}$

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Region (2) : $y \in [1, 4]$

washer $\Rightarrow$ area $= \pi (R^2 - r^2)$

$R \stackrel{\text{Horiz.}}{=} x_R - x_L$

$= \sqrt{y} - (-2)$

$\boxed{R = \sqrt{y} + 2}$

$r \stackrel{\text{Horiz}}{=} x_R - x_L$

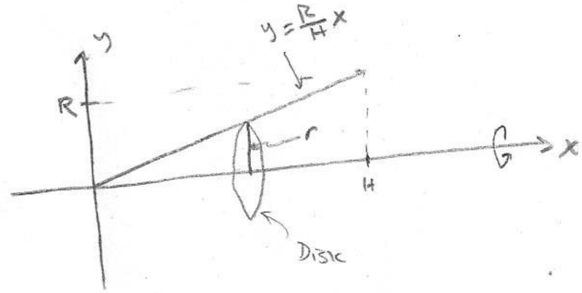
$= y \cancel{-2} - \cancel{(-2)}$

$\boxed{r = y}$

$\Rightarrow \boxed{V_2 = \int_1^4 \pi \left[(\sqrt{y}+2)^2 - (y)^2 \right] dy}$

$\boxed{V = V_1 + V_2}$

(14) Proof: Revolving $R_{xy} = \left\{ y = \frac{R}{H}x,\ x\text{-axis},\ x \in [0, H] \right\}$ about x-axis will create a cone w/ Altitude H & Base Radius R.



Thickness $= dx$

$x \in [0, H]$

Disk $\Rightarrow$ area of Disk $= \pi r^2$

$r \stackrel{\text{vertical}}{=} y_{Top} - y_{bttm}$

$= \frac{R}{H}x - 0$

$V = \int_0^H \pi \left(\frac{R}{H} x \right)^2 dx$

$= \pi \frac{R^2}{H^2} \left[\frac{1}{3} x^3 \right]_0^H$

$= \pi \frac{R^2}{\cancel{H^2}} \frac{1}{3} (H^3)$

$\therefore \boxed{V = \frac{1}{3} \pi R^2 H}$ QED

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(15) Extra credit.

$$I = \int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}}\,dx$$

$$= \int_0^1 \sqrt{1-\left(x^{\frac{1}{2}}\right)^2}\; x^{-\frac{1}{2}}\,dx$$

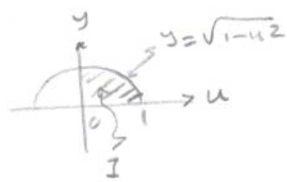
let $u = x^{\frac{1}{2}}$

$2\,du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$\begin{cases} x=1 \Rightarrow u=1 \\ x=0 \Rightarrow u=0 \end{cases}$

$+5$

$$= \int_0^1 \sqrt{1-u^2}\; 2\,du$$



$$= 2\int_0^1 \sqrt{1-u^2}\,du$$

$$= 2\left[\tfrac{1}{4}\pi(1)^2\right]$$

$$\boxed{I = \frac{\pi}{2}}$$