Use **both sides** of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline that was requested for the homework assignments. Also note that you can **only use formulas and methods that were presented** thus far **in the lecture**.

**Differentiate each of the followings:**

**1.** $f(x)=x^{\frac{x}{\ln x}}$ *(Simplify completely)*

**2.** $y=\operatorname{sech}(\ln x)$ *(Simplify completely)*

**3.** $f(x)=\operatorname{arcsec}\sqrt{1+x^2}$ *(Simplify completely)*

**4.** Using calculus definition of natural log, **prove** that $\ln\left[\frac{M}{N}\right]=\ln M-\ln N,\ M>0, N>0$.

**5.** **Prove** $D_x[\operatorname{arccsc} x]=$ ____?____ *(fill-in the blank and prove)*.

**6.** **Prove** the derivative formula for hyperbolic tangent function.

**7.** **Prove** that $D_x\left[\ln|x|\right]=\frac{1}{x}$

**8.** Solve the following differential equation: $\frac{dy}{dx}=x\sqrt{3-4y^2}$.

**Evaluate each of the following integrals:**

**9.** $\int \frac{x+1}{\sqrt{5-3x^2}}dx$

**10.** $\int \sec^5 x\,dx$

**11.** $\int \sin^2 x\cos^4 x\,dx$

**12.** $\int \cos(\ln x)dx$

**13.** $\int \operatorname{sech}(\ln x)dx$

**14.** $\int \frac{1}{\sqrt{3e^{2x}-4}}dx$

15 (Extra Credit) Evaluate $\int \arctan\left(\sqrt{x}\right)dx$.

① $f(x) = x^{\frac{x}{\ln x}}$ ; $x>0,\ x\neq 1$

$\ln[f(x)] = \frac{x}{\cancel{\ln x}} \cancel{\ln x}$ , $x>0, x\neq 1$

$D_x\left( f(x) = e^{x} \right) D_x$

$\boxed{f'(x) = e^{x} \ ;\ x>0,\ x\neq 1}$

(6)

② $y = \operatorname{sech}(\ln x)$

$= \frac{1}{\cosh(\ln x)}$

$= \frac{2}{e^{\ln x} + e^{-\ln x}}$

$= \frac{2}{x+x^{-1}}\left(\frac{x}{x}\right)$

$D_x\left( y = \frac{2x}{x^2+1} \right) D_x$

$\frac{dy}{dx} = 2\ \frac{1(x^2+1) - x(2x)}{(x^2+1)^2}$

$\boxed{\frac{dy}{dx} = \frac{2(1-x^2)}{(x^2+1)^2}}$

(7)

③ $f(x) = \operatorname{arcsec}\sqrt{1+x^2}$

$D_x\left( \ \right) D_x$

$f'(x) = \frac{1}{\sqrt{1+x^2}\sqrt{(\sqrt{1+x^2})^2-1}} \cdot \frac{1}{\cancel{2}}(1+x^2)^{-\frac{1}{2}}(\cancel{2}x)$

$= \frac{1}{\sqrt{1+x^2}\ \sqrt{x^2}} \cdot \frac{x}{\sqrt{1+x^2}}$

$\boxed{f'(x) = \frac{x}{|x|(1+x^2)}}$

(7)

④ prove $\ln\left[\frac{M}{N}\right] = \ln M - \ln N$, $M>0$, $N>0$

Proof: $D_x\left[\ln \frac{x}{N}\right] = \frac{1}{\frac{x}{N}} \cdot \frac{1}{N}$ , $N>0$ , $x>0$

$= \frac{1}{x} \Rightarrow \ln\frac{x}{N}$ is an antideriv. of $\frac{1}{x}$

$D_x[\ln x] = \frac{1}{x} \Rightarrow \ln x$ " " " " "

$\Rightarrow \ln\frac{x}{N}$ & $\ln x$ are antideriv. of $\frac{1}{x}$

$\Rightarrow$ " " " differ by a const.

$\Rightarrow \ln\frac{x}{N} \equiv \ln x + C$ , $\forall x>0$

let $x=N$: $\ln 1 = \ln N + C \Rightarrow \boxed{C = -\ln N}$

$\Rightarrow \ln\frac{x}{N} \equiv \ln x - \ln N$

let $x=M$: $\therefore \boxed{\ln\left(\frac{M}{N}\right) = \ln M - \ln N ,\ M>0, N>0}$ Q.E.D.

(8)

⑤ $D_x[\operatorname{arccsc} x] = \frac{-1}{x\sqrt{x^2-1}}$

Proof: let $y = \operatorname{arccsc} x$ , $\frac{dy}{dx} = ?$

$\Rightarrow \csc y = x$ , $y \in (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

$D_x$ $D_x$

$-\csc y \cot y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{-1}{\csc y \cot y}$

$\csc y = x$ , $y \in Q\,I, III$ , $\cot y = ?$

$\cot y = \pm\sqrt{\csc^2 y - 1}$

$= \pm\sqrt{x^2-1}$

$y \in Q\,I, III \Rightarrow \cot y > 0$

$\left.\right\} \Rightarrow \boxed{\cot y = +\sqrt{x^2-1}}$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

$\therefore \boxed{D_x[\operatorname{arccsc} x] = \frac{-1}{x\sqrt{x^2-1}}}$ Q.E.D.

(8)

(6) $D_x[\tanh x] = \operatorname{sech}^2 x$

Proof: let $y = \tanh x$ , $\frac{dy}{dx} = ?$

$$D_x\left( y = \frac{\sinh x}{\cosh x} \right) D_x$$

$$\frac{dy}{dx} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

7

$$= \frac{1}{\cosh^2 x}$$

$$\frac{dy}{dx} = \operatorname{sech}^2 x$$

$$\therefore \boxed{D_x[\tanh x] = \operatorname{sech}^2 x}$$ Q.E.D

(7) $D_x[\ln|x|] = \frac{1}{x}$

Proof: by case of $x$ ($x>0$ or $x<0$)

Case (1): $x > 0 \Rightarrow |x| = x$

$$D_x[\ln|x|] = D_x[\ln x]$$

$$\boxed{D_x[\ln|x|] = \frac{1}{x}}$$

7

Case (2): $x < 0 \Rightarrow |x| = -x$

$$D_x[\ln|x|] = D_x[\ln(-x)]$$

$$= \frac{1}{-x}(-1)$$

$$\boxed{D_x[\ln|x|] = \frac{1}{x}}$$

by case (1) & (2): $\therefore \boxed{D_x[\ln|x|] = \frac{1}{x}}$ Q.E.D

(8) $\frac{dy}{dx} = x\sqrt{3-4y^2}$

$$\frac{1}{\sqrt{3-4y^2}}\,dy = x\,dx$$

Goal: $\int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u + c$

$$\int \frac{1}{\sqrt{3-4y^2}}\,dy = \int x\,dx$$

7

$$\frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1-\left(\frac{2}{\sqrt{3}}y\right)^2}}\,dy = \frac{1}{2}x^2 + C$$

let $u = \frac{2}{\sqrt{3}} y$

$\frac{\sqrt{3}}{2}\,du = \frac{\cancel{2}}{\sqrt{3}}\,dy$

$$\frac{1}{\cancel{\sqrt{3}}} \frac{\cancel{\sqrt{3}}}{2} \int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}x^2 + C$$

$$\cancel{\tfrac{1}{2}} \arcsin u = \frac{\cancel{1}}{2}x^2 + C$$

$$\arcsin \frac{2}{\sqrt{3}} y = (x^2 + C)$$

$$\frac{2}{\sqrt{3}} y = \sin(x^2 + C) \; ; \; -\frac{\pi}{2} \le x^2 + C \le \frac{\pi}{2} \Rightarrow \boxed{y = \frac{\sqrt{3}}{2}\sin(x^2 + C) \; ; \; -\frac{\pi}{2} \le x^2 + C \le \frac{\pi}{2}}$$

(9) $$I = \int \frac{x+1}{\sqrt{5-3x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{5-3x^2}}\,dx + \int \frac{1}{\sqrt{5-3x^2}}\,dx$$

Goal: $\int u^n\,du = \frac{1}{n+1}u^{n+1} + c$

let $u = 5-3x^2$

$-\frac{1}{6}du = \cancel{(-6)}x\,dx$

(7)

$$= \int (5-3x^2)^{-\frac{1}{2}}\,x\,dx + \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-\left(\frac{\sqrt{3}}{\sqrt{5}}x\right)^2}}\,dx$$

Goal: $\int \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin t + c$

let $t = \frac{\sqrt{3}}{\sqrt{5}}x$

$\frac{\sqrt{5}}{\sqrt{3}}\,dt = \cancel{\frac{\sqrt{3}}{\sqrt{5}}}\,dx$

$$= -\frac{1}{6}\int u^{-\frac{1}{2}}\,du + \frac{1}{\cancel{\sqrt{5}}}\frac{\cancel{\sqrt{5}}}{\sqrt{3}}\int \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= -\frac{1}{6}\,\frac{2}{1}\,u^{\frac{1}{2}} + \frac{1}{\sqrt{3}}\arcsin t + c$$

$$\boxed{I = -\frac{1}{3}\sqrt{5-3x^2} + \frac{1}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{\sqrt{5}}x\right) + C}$$

(10) $$I = \int \sec^5 x\,dx$$

$$= \int \sec^3 x\,\sec^2 x\,dx$$

parts ①: $u = \sec^3 x \xrightarrow{D_x} du = 3\sec^3 x\tan x\,dx$

$v = \tan x \xleftarrow{\int} dv = \sec^2 x\,dx$

$$= \sec^3 x\tan x - 3\int \sec^3 x\,\underbrace{\tan^2 x}_{(\sec^2 x-1)}\,dx$$

(8)

$$= \sec^3 x\tan x \underbrace{- 3\int \sec^5 x\,dx}_{-3I} + 3\int \sec^3 x\,dx$$

$$4I = \sec^3 x\tan x + 3\left[\frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x|\right] + C$$

$$\boxed{I = \frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x + \tan x| + C}$$

(11) $$I = \int \sin^2 x\cos^4 x\,dx$$

$$= \int (\sin x\cos x)^2\cos^2 x\,dx$$

$$= \frac{1}{4}\int \sin^2(2x)\,\frac{1}{2}[1+\cos(2x)]\,dx$$

Goal: $\int u^n\,du = \frac{1}{n+1}u^{n+1} + c$

let $u = \sin(2x)$

$\frac{1}{2}du = \cancel{2}\cos(2x)\,dx$

(7)

$$= \frac{1}{8}\int \sin^2(2x)\,dx + \frac{1}{8}\int \sin^2(2x)\cos(2x)\,dx$$

$$= \frac{1}{8}\,\frac{1}{2}\int [1-\cos(4x)]\,dx + \frac{1}{8}\,\frac{1}{2}\int u^2\,du$$

$$= \frac{1}{16}\int 1\,dx - \frac{1}{16}\int \cos(4x)\,dx + \frac{1}{16}\int u^2\,du$$

$$= \frac{1}{16}x - \frac{1}{16}\,\frac{1}{4}\sin(4x) + \frac{1}{16}\,\frac{1}{3}u^3 + C$$

$$\boxed{I = \frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C}$$

(12) $I = \int \cos(\ln x)\,dx$

Rositás! let $t = \ln x \Rightarrow x = e^t$, $dx = e^t dt$

$I = \int \cos t \; e^t \,dt$

Parts ① $\begin{cases} u = \cos t \xrightarrow{D_t} du = -\sin t\,dt \\ v = e^t \xleftarrow{\int} dv = e^t dt \end{cases}$

$= e^t \cos t + \int \sin t \; e^t \,dt$

Parts ② $\begin{cases} u = \sin t \xrightarrow{D_t} du = \cos t\,dt \\ v = e^t \xleftarrow{\int} dv = e^t dt \end{cases}$

$= e^t \cos t + e^t \sin t - \underbrace{\int \cos t \, e^t dt}_{I}$

$2I = e^t \cos t + e^t \sin t + C$

$$\boxed{I = \frac{1}{2} x \cos(\ln x) + \frac{1}{2} x \sin(\ln x) + C}$$

(7)

(13) $I = \int \operatorname{sech}(\ln x)\,dx$

from exercise # 2 on this exam!

$= \int \frac{2x}{x^2+1}\,dx$

Goal: $\int \frac{1}{u}\,du = \ln|u| + C$

let $u = x^2+1$, $du = 2x\,dx$

$= \int \frac{1}{u}\,du$

$= \ln|u| + C$

$= \ln|x^2+1| + C$

$$\boxed{I = \ln(x^2+1) + C}$$

(6)

(14) $I = \int \frac{1}{\sqrt{3e^{2x}-4}}\,dx$

Goal: $\int \frac{1}{u\sqrt{u^2-1}}\,du = \operatorname{arcsec} u + C$

$= \frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}e^x\right)^2 - 1}}\,dx$

let $u = \frac{\sqrt{3}}{2} e^x \Rightarrow du = \frac{\sqrt{3}}{2} e^x\,dx$

$= \frac{1}{2} \int \frac{1}{\left(\frac{\sqrt{3}}{2}e^x\right)\sqrt{\left(\frac{\sqrt{3}}{2}e^x\right)^2 - 1}} \left(\frac{\sqrt{3}}{2}e^x\right) dx$

$= \frac{1}{2} \int \frac{1}{u\sqrt{u^2-1}}\,du$

$= \frac{1}{2} \operatorname{arcsec} u + C$

$$\boxed{I = \frac{1}{2} \operatorname{arcsec}\left(\frac{\sqrt{3}}{2} e^x\right) + C}$$

(8)

(15) (ExTra Credit)

$I = \int \arctan\sqrt{x}\, dx$

Rosita's! let $t = \arctan\sqrt{x}$

$\tan t = \sqrt{x}$ , $\underline{-\frac{\pi}{2} < t < \frac{\pi}{2}}$

$\Rightarrow x = \tan^2 t$

$dx = 2\tan t \sec^2 t\, dt$

$I = \int t\, 2\tan t \sec^2 t\, dt$

Parts ① $\begin{cases} u = t \xrightarrow{D_t} du = dt \\ v = \tan^2 t \xleftarrow{\int} dv = 2\tan t \sec^2 t\, dt \end{cases}$

$= t\tan^2 t - \int \underbrace{\tan^2 t}_{(\sec^2 t - 1)}\, dt$

$= t\tan^2 t - \tan t + t + C$

$\boxed{I = (\arctan\sqrt{x})(x) - (\sqrt{x}) + (\arctan\sqrt{x}) + C}$

+5