Math 8 **Exam #2** **Name:** Key

Use **both sides** of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline that was requested for the homework assignments. Also note that you can **only use formulas and methods that were presented** thus far **in the lecture**.

Evaluate each of the following integrals:

1. $\int \frac{1}{x^2\sqrt{4x^2-9}}dx$

2. $\int \frac{1}{x^3-4x^2}dx$

3. $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}dx$

4. $\int \frac{1}{x^2+x+1}dx$

Prove convergence or divergence for each of the followings. If convergence, find the corresponding sum or limit accordingly:

5. $\left\{\cos(n\pi)\frac{(-1)^n n^2}{3n^2+1}\right\}$

6. $\left\{\left(1+\frac{2}{3n}\right)^{4n}\right\}$

7. $\sum_{n=0}^{\infty}(-1)^n\frac{3^{n-1}}{2^{2n+1}}$

8. $\sum_{n=2}^{\infty}\ln\left(\frac{n+1}{n-1}\right)$

9. Prove the Geometric Series Theorem ($\sum_{n=1}^{\infty}a_1r^{n-1}$).

10. Prove convergence or divergence of the following: $\int_1^{\infty}\frac{1}{x^p}dx$ for $p>1$.

11. Use the template for *e* to evaluate: $\lim_{x\to-\infty}\left(1-\frac{2}{x}\right)^{3x}$

12. (Extra Credit) Evaluate the following integral: $\int \frac{1}{ax^n+bx}dx$

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(1) $I = \int \frac{1}{x^2\sqrt{4x^2-9}}\,dx$

$= \frac{1}{3}\int \frac{1}{x^2\sqrt{\left(\frac{2x}{3}\right)^2-1}}\,dx$ $\qquad \frac{2x}{3} = \sec\theta \Longleftarrow \theta = \operatorname{arcsec}\left(\frac{2x}{3}\right)$

$\theta \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

$x = \frac{3}{2}\sec\theta$

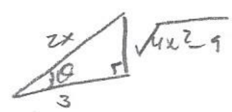
$= \frac{1}{3}\int \frac{1}{\frac{9}{4}\sec^{\cancel{2}}\theta\,\underbrace{\sqrt{\sec^2\theta-1}}_{\tan^2\theta}}\,\frac{3}{2}\cancel{\sec\theta}\tan\theta\,d\theta$ $\qquad dx = \frac{3}{2}\sec\theta\tan\theta\,d\theta$ (12)

$= \frac{1}{\cancel{3}}\frac{4^2}{9}\frac{\cancel{3}}{2}\int \frac{1}{\sec\theta\,|\tan\theta|}\tan\theta\,d\theta$, $\quad \theta \in QI, III \Rightarrow \tan\theta > 0 \Rightarrow |\tan\theta| = \tan\theta$

$= \frac{2}{9}\int \frac{1}{\sec\theta\,\cancel{\tan\theta}}\cancel{\tan\theta}\,d\theta$

$= \frac{2}{9}\int \cos\theta\,d\theta$

$= \frac{2}{9}\sin\theta + C$ $\qquad \sec\theta = \frac{2x}{3}$



$= \frac{\cancel{2}}{9}\frac{\sqrt{4x^2-9}}{\cancel{2}x} + C$

$\boxed{I = \frac{1}{9}\frac{\sqrt{4x^2-9}}{x} + C}$

(2) $I = \int \frac{1}{x^3-4x^2}\,dx$

PFD: $\frac{1}{x^2(x-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$

$1 \equiv Ax(x-4) + B(x-4) + Cx^2, \ \forall x$

~~let~~ $x=0 \Rightarrow 1 = -4B \Rightarrow \boxed{B = \frac{-1}{4}}$

$x=4 \Rightarrow 1 = 16C \Rightarrow \boxed{C = \frac{1}{16}}$

$x=1 \Rightarrow 1 = A(-3) - \frac{1}{4}(-3) + \frac{1}{16} \Rightarrow \boxed{A = -\frac{1}{16}}$ (12)

$I = -\frac{1}{16}\int \frac{1}{x}\,dx - \frac{1}{4}\int x^{-2}\,dx + \frac{1}{16}\int \frac{1}{x-4}\,dx$

$= -\frac{1}{16}\ln|x| - \frac{1}{4}(-1)x^{-1} + \frac{1}{16}\ln|x-4| + C$

$\boxed{I = -\frac{1}{16}\ln|x| + \frac{1}{4x} + \frac{1}{16}\ln|x-4| + C}$

(3) $I = \int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx$

$= \int \frac{1}{x^{\frac{3}{6}} - x^{\frac{2}{6}}}\,dx$ let $u = x^{\frac{1}{6}} \Rightarrow x = u^6$, $dx = 6u^5\,du$

$= \int \frac{1}{u^3 - u^2}\,6u^5\,du$

$= 6 \int \frac{u^3}{u-1}\,du$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & 0 & 0 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 \end{array}$$

(10)

$= 6 \int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du$

$= 6\left[\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right] + C$

$= 2x^{\frac{3}{6}} + 3x^{\frac{2}{6}} + 6x^{\frac{1}{6}} + 6\ln|x^{\frac{1}{6}} - 1| + C$

$\boxed{I = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C}$

(4) $I = \int \frac{1}{x^2 + x + 1}\,dx$

Compl. □: $x^2 + x + 1 = (x^2 + 1x) + 1$
$= (x^2 + 1x + \frac{1}{4}) + 1 - \frac{1}{4}$
$= (x + \frac{1}{2})^2 + \frac{3}{4}$

$= \int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}}\,dx$ let $t = x + \frac{1}{2}$, $dt = dx$

$= \int \frac{1}{t^2 + \frac{3}{4}}\,dt$

$= \frac{4}{3} \int \frac{1}{(\frac{2}{\sqrt{3}}t)^2 + 1}\,dt$ let $u = \frac{2}{\sqrt{3}}t$, $\frac{\sqrt{3}}{2}\,du = dt$ ($du = \frac{2}{\sqrt{3}}dt$)

(10)

$= \frac{\cancel{4}^{2}}{3} \frac{\sqrt{3}}{\cancel{2}} \int \frac{1}{u^2 + 1}\,du$

$= \frac{2\sqrt{3}}{3} \arctan u + C$

$= \frac{2\sqrt{3}}{3} \arctan\left(\frac{2}{\sqrt{3}}t\right) + C$

$\boxed{I = \frac{2\sqrt{3}}{3} \arctan\left[\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right] + C}$

(5) $\left\{ \cos(n\pi) \frac{(-1)^n n^2}{3n^2+1} \right\}$

Proof: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n\pi) \frac{(-1)^n n^2}{3n^2+1}$

$= \lim_{n\to\infty} \cancel{(-1)^n}(-1)^n \frac{\cancel{n^2}\ 1}{\cancel{n^2}(3+\frac{1}{n^2})}$

$\boxed{\lim_{n\to\infty} a_n = \frac{1}{3}}$

$\therefore$ $\boxed{\text{the seq. } \left\{ \cos(n\pi) \frac{(-1)^n n^2}{3n^2+1} \right\} \text{ converges to } \frac{1}{3}}$

(8)

(6) $\left\{ \left(1+\frac{2}{3n}\right)^{4n} \right\}$

Proof: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{2}{3n}\right)^{4n}$

$= \lim_{n\to\infty} \left(1+\frac{1}{\frac{3n}{2}}\right)^{4n}$ let $u=\frac{3}{2}n \Rightarrow n=\frac{2}{3}u$

as $n\to\infty \Rightarrow u\to\infty$

$= \lim_{u\to\infty} \left(1+\frac{1}{u}\right)^{4(\frac{2}{3})u}$

$= \lim_{u\to\infty} \left[\left(1+\frac{1}{u}\right)^u\right]^{\frac{8}{3}}$

$\boxed{\lim_{n\to\infty} a_n = e^{\frac{8}{3}}}$

$\therefore$ $\boxed{\text{The seq. } \left\{ \left(1+\frac{2}{3n}\right)^{4n} \right\} \text{ converges to } e^{\frac{8}{3}}}$

(8)

(7) $\sum_{n=0}^{\infty} (-1)^n \frac{3^{n-1}}{2^{2n+1}}$

Proof: $\sum_{n=0}^{\infty} (-1)^n \frac{3^{n-1}}{2^{2n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{3^n \cdot 3^{-1}}{2^{2n} \cdot 2}$

$= \frac{1}{6} \sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n$ is a Geom. Series w/ $r=-\frac{3}{4} \Rightarrow |r| = \frac{3}{4} < 1$

$\Rightarrow$ Geom. Series Converges to $S = \frac{a_1}{1-r}$, $a_1 = \frac{1}{6}$, $r = -\frac{3}{4}$ $\Rightarrow S = \frac{\frac{1}{6}}{1+\frac{3}{4}}$

$S = \frac{2}{21}$

$\therefore$ $\boxed{\text{The Series } \sum_{n=0}^{\infty} (-1)^n \frac{3^{n-1}}{2^{2n+1}} \text{ converges to } S = \frac{2}{21}}$

(8)

(8) $\sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n-1}\right)$

Proof: ① $S_p = \sum_{n=2}^{p} [\ln(n+1) - \ln(n-1)]$

$= (\cancel{\ln 3} - \ln 1) + (\cancel{\ln 4} - \ln 2) + (\cancel{\ln 5} - \cancel{\ln 3}) + (\ln 6 - \cancel{\ln 4}) + (\ln 7 - \cancel{\ln 5}) + \cdots + (\ln(p) - \cancel{\ln(p-2)}) + (\ln(p+1) - \cancel{\ln(p-1)})$

$\boxed{S_p = -\ln 1 - \ln 2 + \cdots + \ln p + \ln(p+1)}$

② $\lim_{p\to\infty} S_p = \lim_{p\to\infty} [-\ln 2 + \cdots + \ln p + \ln(p+1)]$

$= \infty$

$\therefore$ $\boxed{\text{The series } \sum \ln\left(\frac{n+1}{n-1}\right) \text{ Diverges}}$

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(9) Prove Geom. Series Thm:

Proof: Given Geom. Series $\sum a_1 r^{n-1}$

$S_p = \sum_{n=1}^{p} a_1 r^{n-1}$

$S_p = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{p-1}$, find S_p

$-rS_p = \quad -a_1 r - a_1 r^2 - \cdots - a_1 r^{p-1} - a_1 r^p$

$S_p - rS_p = a_1 - a_1 r^p$

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$S_p(1-r) = a_1 - a_1 r^p$

$\boxed{S_p = \frac{a_1}{1-r} - \frac{a_1}{1-r} r^p}$

Case ①: $|r|<1 \Rightarrow \lim_{p\to\infty} r^p = 0$

$\Rightarrow \boxed{\lim_{p\to\infty} S_p = \frac{a_1}{1-r}} \Rightarrow \boxed{\text{Geom. series converges to } S = \frac{a_1}{1-r} \text{ when } |r|<1}$

Case ② $r>1 \Rightarrow \lim_{p\to\infty} r^p = \infty$

$\Rightarrow \lim_{p\to\infty} S_p = -\frac{a_1}{1-r}(\infty) \Rightarrow$ $\boxed{\text{Geom. series Diverges when } r>1}$

Case ③ $r<-1 \Rightarrow \lim_{p\to\infty} r^p$ DNE

$\Rightarrow \lim_{p\to\infty} S_p$ DNE $\Rightarrow$ $\boxed{\text{'' \quad '' \quad '' \quad '' } r<-1}$

$\therefore$ $\boxed{\text{Geom. Series } \sum a_1 r^{n-1} \begin{cases} \text{Converges to } S = \frac{a_1}{1-r} & \text{when } |r|<1 \\ \text{Diverges} & \text{when } |r|>1 \end{cases}}$ Q.E.D.

(10) $\int_1^\infty \frac{1}{x^P}\,dx$, $P>1$

Proof: $I = \int_1^\infty \frac{1}{x^P}\,dx$, $P>1$

$= \lim_{M\to\infty} \int_1^M x^{-P}\,dx$, $P>1$

$= \lim_{M\to\infty} \left[\frac{1}{-P+1} x^{-P+1}\right]_1^M$; $P>1$

$= \frac{1}{-P+1} \lim_{M\to\infty} \left[M^{-P+1} - 1\right]$; $P>1 \Rightarrow -P<-1 \Rightarrow -P+1<0$

inv. $\infty^{-P+1} \to 0$ -1

$\boxed{I = \frac{-1}{-P+1}}$

$\therefore$ $\boxed{\text{The integral } \int_1^\infty \frac{1}{x^P}dx \text{ converges to } \frac{1}{P-1}}$

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(11) $\lim_{x\to-\infty} \left(1-\frac{2}{x}\right)^{3x} = \lim_{x\to-\infty}\left[1+\frac{1}{\left(\frac{-x}{2}\right)}\right]^{3x}$ $\left\{\begin{array}{l}\text{let } u=-\frac{x}{2} \Rightarrow x=-2u \\ \text{as } x\to-\infty \Rightarrow u\to+\infty\end{array}\right.$

$= \lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{3(-2u)}$

$= \lim_{u\to\infty}\left[\left(1+\frac{1}{u}\right)^{u}\right]^{-6}$

$\boxed{\lim_{x\to-\infty}\left(1-\frac{2}{x}\right)^{3x} = e^{-6}}$

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(12) (Extra Credit) $I = \int \frac{1}{ax^n+bx}\,dx$

$= \int \frac{1}{x^n(a+bx^{-n+1})}\,dx$

$= \int \frac{1}{a+bx^{-n+1}}\, x^{-n}\,dx$ let $u = a+bx^{-n+1}$

$\frac{1}{b(-n+1)}du = b(-n+1)x^{-n}\,dx$

$= \frac{1}{b(1-n)}\int \frac{1}{u}\,du$

$= \frac{1}{b(1-n)} \ln|u| + C$

$\boxed{I = \frac{1}{b(1-n)} \ln\left|a+bx^{-n+1}\right| + C}$

$\boxed{+5}$