Math 8 **Exam #3** **Name:** Key

Use **both sides** of the provided blank sheets of paper for your work. Throughout this exam, it is essential to **show details of your work to *support* your answer**. Merely stating the final answer will result in no credit. Follow the same format guideline that was requested for the homework assignments. Also note that you can **only use formulas and methods that were presented** thus far **in the lecture**.

1. Prove the Harmonic Series Theorem.

2. Use the Integral test to prove convergence or divergence for $\sum \frac{\arctan n}{n^2+1}$.

3. Prove convergence or divergence for the following sequence. If convergent, find its limit: $\left\{\frac{n^n}{n!}\right\}$.

Prove convergence or divergence of each of the followings using the stated test:

4. $\sum \frac{n^2}{n^3+1}$ (Using BCT only)

5. $\sum \frac{\ln n}{n^2}$ (using any test of your choice)

Prove absolute convergence, conditional convergence or divergence for each of the followings:

6. $\sum \cos(n\pi)\left(\frac{n}{n+1}\right)^n$

7. $\sum \frac{\sin n}{n^2+1}$

8. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-1}$

Find the interval, radius and center of convergence for each of the followings:

9. $\sum \frac{(-1)^n}{n2^n}(3x-1)^n$

10. $\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!}(3x+1)^n$

11. Use the known power series for $\frac{1}{1-x}$, to find a power series representation for $f(x)=\frac{x^2+1}{3x+1}$ (note, that your answer must be a <u>single</u> power series).

12. Find the Taylor Series representation of $f(x)=\frac{1}{x}$ centered about "2".

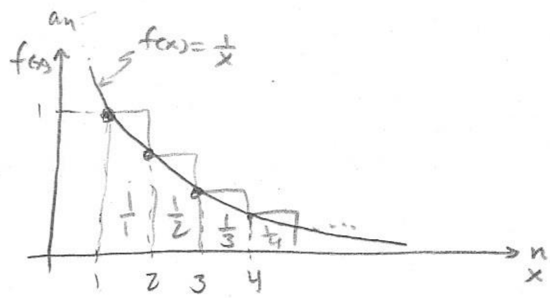
13. **(Extra Credit)** Find the exact sum for the following convergent series: $\sum_{n=2}^{\infty} \frac{\cos(n\pi)2^{2n+1}}{3^{2n-1}(2n)!}$.

① Thm: $\sum \frac{1}{n}$ Diverges

Proof: Given $\sum \frac{1}{n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

let $f(x) = \frac{1}{x}$

consider Graphically,



from thee Graph:

(8)

sum of areas of rectangles $= \sum \frac{1}{n}$ is an over-estimation of $\int_1^\infty \frac{1}{x}\,dx$

$$\left.\begin{array}{l} \Rightarrow \int_1^\infty \frac{1}{x}\,dx < \sum \frac{1}{n} \\ \text{by a Thm: } \int_1^\infty \frac{1}{x}\,dx = \infty \end{array}\right\} \Rightarrow \sum \frac{1}{n} = \infty$$

$\therefore$ $\boxed{\text{The Harmonic Series } \sum \frac{1}{n} \text{ Diverges}}$ Q.E.D

② $\sum \frac{\arctan n}{n^2+1}$

Proof; Integral test: requirement:

① $a_n = \frac{\arctan n}{n^2+1} > 0 \;\; \forall n \geq 1$ ✓

② $f(x) = \frac{\arctan x}{x^2+1} \Rightarrow f'(x) = \frac{1-2x\arctan x}{(x^2+1)^2}$

(10)

Given $x \geq 1 \Rightarrow \arctan x \geq \arctan 1 = \frac{\pi}{4}$

$\Rightarrow 2\arctan x \geq \frac{\pi}{2} > 1 \Rightarrow$ $2\arctan x > 1$

$x > 1$

$2x\arctan x > 1$

$-2x\arctan x < -1$

$$\left.\begin{array}{l} 1-2x\arctan x < 0 \\ (x^2+1)^2 > 0 \end{array}\right\} \Rightarrow f'(x) = \frac{1-2x\arctan x}{(x^2+1)^2} < 0 \;\; \forall x \geq 1$$

$\Rightarrow$ $\boxed{f(x) \text{ is mono. dec.}}$ ✓

$I = \int_1^\infty \frac{\arctan x}{x^2+1}\,dx$

$= \lim_{M\to\infty} \int_1^M (\arctan x)\frac{1}{x^2+1}\,dx$

$= \lim_{M\to\infty} \left[\frac{1}{2}(\arctan x)^2\right]_1^M$

$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] \Rightarrow$ $\boxed{\text{Integral } I \text{ is Convergent}}$

$\therefore$ $\boxed{\text{the series } \sum \frac{\arctan n}{n^2+1} \text{ Converges by Integral Test.}}$

(3) $\left\{\frac{n^n}{n!}\right\}$

Proof: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^n}{n!}$ stuck! $\qquad$ inv. $\frac{\infty}{\infty}$ indet.

Consider $\sum \frac{n^n}{n!}$

Ratio test: req.: $a_n > 0$ ✓

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{\cancel{(n+1)!}} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$= e > 1 \xrightarrow[\text{Thm}]{\text{by Bonus}} \boxed{\lim_{n\to\infty} a_n = \infty}$$

(8)

$\therefore$ $\boxed{\text{The seq. } \left\{\frac{n^n}{n!}\right\} \text{ Diverges}}$

(4) $\sum \frac{n^2}{n^3+1}$ (B.C.T)

(10)

Proof: B.C.T. req. $a_n > 0$ ✓

Pick $b_n = \frac{1}{2n}$

Given $1 \le n$

$\Rightarrow 1 \le n^3$

$\quad +n^3 \ \ +n^3$

$\Rightarrow n^3+1 \le 2n^3$

$\Rightarrow \frac{1}{n^3+1} \geq \frac{1}{2n^3}$

$\Rightarrow \frac{n^2}{n^3+1} \geq \frac{n^2}{2n^3} = \frac{1}{2n}$

$\boxed{a_n \geq b_n}$ (1)

$\sum b_n = \sum \frac{1}{2}\frac{1}{n}$

$= \frac{1}{2}\sum \frac{1}{n}$ is a Harmonic Series

$\Rightarrow$ $\boxed{\sum b_n \text{ Diverges}}$ (2)

by (1) & (2):

$\therefore$ $\boxed{\text{the series } \sum \frac{n^2}{n^3+1} \text{ Diverges by B.C.T.}}$

Guess:

a_n

$\frac{n^2}{n^3+1} \longrightarrow \frac{n^2}{n^3} = \frac{1}{n}$ Guess Divergent

WNT $a_n \geq b_n$

$\frac{n^2}{n^3+1} \overset{?}{\geq} \frac{\cancel{n^2}}{n^3}$

$n^3+1 \overset{?}{\le} n^3$

$1 \overset{?}{\le} 0$ False!

change b_n to $\frac{1}{2}\frac{n^2}{n^3}$

$\frac{\cancel{n^2}}{n^3+1} \overset{?}{\geq} \frac{\cancel{n^2}}{2n^3}$

$n^3+1 \overset{?}{\le} 2n^3$

$1 \le n^3$

$1 \le n$ ✓

(5) $\sum \frac{\ln n}{n^2}$

Guess $\frac{\ln n}{n^2} \longrightarrow \frac{1}{n^2}$ Converget.

Proof: L.C.T. $a_n = \frac{\ln n}{n^2} > 0 \ \forall n \geq 2$ ✓

pick $b_n = \frac{1}{n^{\frac{3}{2}}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{\frac{3}{2}}}}$$

$$= \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{2}}} \qquad \text{inv. } \frac{\infty}{\infty} \text{ indet.}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2} n^{-\frac{1}{2}}}$$

$$= \lim_{n\to\infty} \frac{2}{n^{\frac{1}{2}}}$$

$$= 0 \Longrightarrow a_n << b_n$$

$\sum b_n = \sum \frac{1}{n^{\frac{3}{2}}}$ is a p-series w/ $p = \frac{3}{2} > 1 \Rightarrow \sum b_n$ converges.

$\therefore$ $\boxed{\text{The series } \sum \frac{\ln n}{n^2} \text{ converges by L.C.T.}}$

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(6) $\sum \cos(n\pi)\left(\frac{n}{n+1}\right)^n$

Proof: Div. Test: $\sum \cos(n\pi)\left(\frac{n}{n+1}\right)^n = \sum (-1)^n \left(\frac{n}{n+1}\right)^n$

$$\lim_{n\to\infty} |c_n| = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \left(\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n}$$

$$= \frac{1}{e} \neq 0$$

$\therefore$ $\boxed{\text{the series } \sum \cos(n\pi)\left(\frac{n}{n+1}\right)^n \text{ Diverges by Div. test}}$

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(7) $\sum \frac{\sin n}{n^2+1}$

Proof: $\sum \frac{\sin n}{n^2+1}$ is not a pos.-Term Series (NOT Alt. series either)

Consider $\sum |c_n| = \sum \frac{|\sin n|}{n^2+1}$

B.C.T. req. $|c_n| > 0$ ✓

(10)

Pick $b_n = \frac{1}{n^2}$

$$\begin{array}{c} 1 > 0 \\ +n^2 \quad +n^2 \\ \hline n^2+1 > n^2 \end{array}$$

$$\left.\begin{array}{l} 0 < \frac{1}{n^2+1} < \frac{1}{n^2} \\ 0 < |\sin n| \leq 1 \end{array}\right\} \xrightarrow[\text{side by side}]{\text{mult.}} \frac{|\sin n|}{n^2+1} < \frac{1}{n^2}$$

$\Rightarrow \boxed{|c_n| < b_n}$ (1)

$\sum b_n = \sum \frac{1}{n^2}$ is a p-series w/ $p = 2 > 1 \Rightarrow \boxed{\sum b_n \text{ converges}}$ (2)

by (1) & (2):

$\Rightarrow \sum |c_n|$ converges by B.C.T.

$\therefore$ $\boxed{\text{the series } \sum \frac{\sin n}{n^2+1} \text{ is Abs. convergent.}}$

(8) $\sum_{n=2}^{\infty} (-1)^n \frac{n}{n^2-1}$

Proof: Consider $\sum |c_n| = \sum \frac{n}{n^2-1}$

(10)

L.C.T. $|c_n| > 0$ ✓, Pick $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2-1} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}(1-\frac{1}{n^2})}$$

$$= 1$$

$\sum b_n = \sum \frac{1}{n}$ is harmonic series $\Rightarrow \sum b_n$ Diverges. $\Rightarrow \boxed{\sum |c_n| \text{ Diverges by L.C.T.}}$ (1)

Sorry for making this tight!

A.S.T.: (1) $\sum (-1)^n \frac{n}{n^2-1}$ is Alt. series ✓

(2) $\lim_{n\to\infty} |c_n| = \lim_{n\to\infty} \frac{n}{n^2(1-\frac{1}{n^2})}$

$= 0$ ✓

inv.: $\frac{1}{\infty(1-\frac{1}{\infty})} \to 0$

(3) $f(x) = \frac{x}{x^2-1} \Rightarrow f'(x) = \frac{x^2-1-x(2x)}{(x^2-1)^2} \Rightarrow f'(x) = -\frac{x^2+1}{(x^2-1)^2}$

$$\left.\begin{array}{l} x^2+1 > 0 \\ (x^2-1)^2 > 0 \end{array}\right\} \Rightarrow \frac{x^2+1}{(x^2-1)^2} > 0 \Rightarrow f'(x) = -\frac{x^2}{(x^2-1)^2} < 0 \Rightarrow \boxed{f(x) \text{ is mono. dec.}}$$

$\Rightarrow a_n$ is mono. dec. ✓

$\Rightarrow \boxed{\sum c_n \text{ Converges by A.S.T.}}$ (2) by (1) & (2): $\therefore$ $\boxed{\text{The series } \sum (-1)^n \frac{n}{n^2-1} \text{ is Cond. Convergent}}$

(9) $\sum \frac{(-1)^n}{n\,2^n}(3x-1)^n = \sum (-1)^n \frac{3^n}{n\,2^n}\left(x-\frac{1}{3}\right)^n \Rightarrow \boxed{C=\frac{1}{3}}$

① Ratio test. $\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}\left(x-\frac{1}{3}\right)^{n+1}}{(n+1)\,2^{n+1}}\cdot\frac{n\,\cancel{2^n}}{\cancel{3^n}\cancel{\left(x-\frac{1}{3}\right)^n}}\right|$

$= \frac{3}{2}\left|x-\frac{1}{3}\right| \underbrace{\lim_{n\to\infty}\frac{n}{n+1}}_{\to 1}$

$= \frac{3}{2}\left|x-\frac{1}{3}\right|$

for convergence: $\frac{3}{2}\left|x-\frac{1}{3}\right|<1$

$\left|x-\frac{1}{3}\right|<\frac{2}{3} \Rightarrow \boxed{R=\frac{2}{3}}$

$-\frac{2}{3}<x-\frac{1}{3}<\frac{2}{3}$

$-\frac{1}{3}<x<1$

② Check $x=-\frac{1}{3}$: $\sum (-1)^n\frac{3^n}{n\,2^n}\left(x-\frac{1}{3}\right)^n\Big|_{x=-\frac{1}{3}} = \sum (-1)^n\frac{3^n}{n\,2^n}\left(-\frac{2}{3}\right)^n$

$= \sum \frac{1}{n}$ is harmonic $\Rightarrow$ Divergent.

$\Rightarrow \boxed{x \neq -\frac{1}{3}}$

③ Check $x=1$: $\sum (-1)^n\frac{3^n}{n\,2^n}\left(x-\frac{1}{3}\right)^n\Big|_{x=1} = \sum (-1)^n\frac{1}{n}$ is Alt. Harmonic $\Rightarrow$ Convergent

$\Rightarrow \boxed{x=1}$ ✓

$\therefore \boxed{I.C. = \left(-\frac{1}{3}, 1\right],\ C=\frac{1}{3},\ R=\frac{2}{3}}$

(8)

(10) $\sum_{n=0}^{\infty}\frac{2^{n+1}}{n!}(3x+1)^n = \sum_{n=0}^{\infty}\frac{2^{n+1}3^n}{n!}\left(x+\frac{1}{3}\right)^n \Rightarrow \boxed{C=-\frac{1}{3}}$

Ratio test: $\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+2}\,3^{n+1}\left(x+\frac{1}{3}\right)^{n+1}}{(n+1)!}\cdot\frac{n!}{2^{n+1}3^n\left(x+\frac{1}{3}\right)^n}\right|$

$=6\left|x+\frac{1}{3}\right|\underbrace{\lim_{n\to\infty}\frac{1}{n+1}}_{\to 0}$

$=0<1 \quad \forall\ x \Rightarrow$ Convergent $\forall x\in\mathbb{R}$

$\therefore \boxed{I.C. = (-\infty,\infty),\ R=\infty,\ C=-\frac{1}{3}}$

(6)

(11) $f(x) = \dfrac{x^2+1}{3x+1}$ Long Div.

$$\begin{array}{r} \frac{1}{3}x - \frac{1}{9} \\ 3x+1 \overline{) x^2 + 1} \\ \underline{\ominus x^2 \ominus \frac{1}{3}x} \\ -\frac{1}{3}x + 1 \\ \underline{\oplus -\frac{1}{3}x \oplus -\frac{1}{9}} \\ \frac{10}{9} \end{array}$$

$$f(x) = \frac{-1}{9} + \frac{1}{3}x + \frac{10}{9}\,\frac{1}{3x+1}$$

$$= -\frac{1}{9} + \frac{1}{3}x + \frac{10}{3^2}\,\frac{1}{1-(-3x)}$$

$$= -\frac{1}{9} + \frac{1}{3}x + \frac{10}{3^2}\sum_{n=0}^{\infty}(-3x)^n \quad ; \quad |-3x| < 1 \Rightarrow 3|x| < 1$$

$$\boxed{f(x) = -\frac{1}{9} + \frac{1}{3}x + 10\sum_{n=0}^{\infty}(-1)^n\, 3^{n-2}\, x^n \; ; \qquad |x| < \frac{1}{3}}$$

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(12) Taylor Series $f(x) = \frac{1}{x}$ $c = 2$

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$\frac{f^{(n)}(2)}{n!}(x-2)^n$
0	x^{-1}		
1	$-1x^{-2}$		
2	$+(1)(2)\,x^{-3}$		
3	$-(1)(2)(3)\,x^{-4}$		
$\vdots$	$\vdots$		
n	$(-1)^n\, n!\, x^{-(n+1)}$	$(-1)^n\, n!\, 2^{-(n+1)}$	$\frac{(-1)^n \cancel{n!}\, 2^{-(n+1)}}{\cancel{n!}}(x-2)^n$

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$$\boxed{f(x) = \sum_{n=0}^{\infty}(-1)^n\,\frac{1}{2^{(n+1)}}\,(x-2)^n}$$

Extra credit

(13) $$\sum_{n=2}^{\infty}\frac{\cos(n\pi)\, 2^{2n+1}}{3^{2n-1}(2n)!} = \sum_{n=2}^{\infty}\frac{(-1)^n\, 2^{2n}\cdot 2^1}{(2n)!\cdot 3^{2n}\cdot 3^{-1}}$$

+5

$$= 6\sum_{n=2}^{\infty}(-1)^n\frac{1}{(2n)!}\left(\frac{2}{3}\right)^{2n}$$

n=0 n=1

$$= 6\left[\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(\frac{2}{3}\right)^{2n}\right) - \left(1 - \frac{1}{2}\left(\frac{2}{3}\right)^2\right)\right]$$

$$\Rightarrow \boxed{S = 6\left[\cos\left(\frac{2}{3}\right) - \frac{7}{9}\right]}$$